11-15-68

ERIC REPORT RESUME

ERIC ACC. NO.				
ED 034 087				
CH ACC. NO.	P.A.	PUBL. DATE	ISSUE	IS DOCUMENT COPYRIGHTED? YES ☐ NO ☒
AA 000 447		70	RIEAPR70	ERIC REPRODUCTION RELEASE? YES ☐ NO ☒
				LEVEL OF AVAILABILITY I ☒ II ☐ III ☐

AUTHOR
Suydam, Marilyn; Riedesel, C. Alan

TITLE
Research on Elementary Mathematics. PREP XI.

SOURCE CODE	INSTITUTION (SOURCE)
SYN71840	Pennsylvania State Univ., University Park.
SP. AG. CODE	SPONSORING AGENCY
RMQ66000	Office of Education (DHEW), Washington, D.C.

EDRS PRICE	CONTRACT NO.	GRANT NO.
0.50;3.20		
REPORT NO.		BUREAU NO.
PREP-11		

AVAILABILITY

JOURNAL CITATION

DESCRIPTIVE NOTE
62p.

DESCRIPTORS
*Elementary School Mathematics; *Research; Elementary School Teachers; Mathematics Instruction; Mathematical Concepts;*Modern Mathematics; Measurement;*Readiness (Mental)

IDENTIFIERS

ABSTRACT
This study interprets and evaluates elementary matehmatics research literature and current teaching practice. The list contains three related documents: (11-A) gives answers from research on elementary mathematics. It is the authors synthesis and interpretation of research findings in question-and-answer format; (11-B) offers generalizations and implications on elementary school mathematics which the authors believe to be clearly substantiated by research; (11-C) lists current research documents on elementary mathematics available from ERIC Document Reproduction Service. This project was funded by the Office of Education under its Targated Communications Program. (ON)

ED 034 087

PREP

RESEARCH ON ELEMENTARY MATHEMATICS

No. 11

PREP is . . .

- *a synthesis and interpretation of research, development, and current practice on a specific educational topic*
- *a method of getting significant R&D findings to the practitioner quickly*
- *the best thinking of researchers interpreted by specialists in simple language*
- *the focus of research on current educational problems*
- *a format which can be easily and inexpensively reproduced for wide distribution*
- *raw material in the public domain which can be adapted to meet local needs*
- *an attempt to improve our Nation's schools through research*

P *utting*
R *esearch into*
E *ducational*
P *ractice*

AA 000 447

Research offers some answers to questions frequently asked by educators, parents, and textbook publishers on elementary mathematics, noted Dr. Marilyn Suydam and Dr. C. Alan Riedesel of the Pennsylvania State University, University Park, after an extensive review of these audiences' needs and a comprehensive survey of the elementary mathematics research literature and current practice. This project was funded by the Office of Education under its Targeted Communications Program.

Document 11-A, in a question-and-answer format, is their synthesis and interpretation of these research findings. Only findings evaluated as valid are cited, unless limitations indicate otherwise. Bibliographic information on the research documents cited is not included in this kit because of its length; however, it is available (along with abstracts of these documents) in one of the volumes of the final report of the project, which may be obtained from the ERIC Document Reproduction Service (EDRS), The National Cash Register Company, 4936 Fairmont Avenue, Bethesda, Maryland 20014 under the identification number ED 030 017 at a cost of $1.25 for microfiche and $16.65 for hard copy.

Also included in another volume of the final report (but again not in this kit because of their length) are interviews, conducted by Dr. Suydam and Dr. Riedesel, with the directors of 10 of the major curriculum developmental mathematics projects. Discussed in these interviews are the background, objectives, activities, accomplishments/failures, evaluation, and future directions of these projects. The 10 projects and the directors interviewed are:

Projet Mathematique de Sherbrooke--Zoltan P. Dienes

African Math Program--Hugh P. Bradley

Cambridge Conference on School Mathematics--Hugh P. Bradley

Stanford Computer-Assisted Instruction Projects--Patrick Suppes

School Mathematics Study Group (SMSG)--E. G. Begle

Individually Prescribed Instruction (RBS)--Robert Scanlon

The Madison Project--Robert Davis

Comprehensive School Mathematics Program (CSMP/CEMREL--
Burt Kaufman

Minnemast (Minnesota School Mathematics and Science Project)--
Roger Jones

University of Geneva--Jean Piaget and Barbel Inhelder

The volume containing these interviews is also available from EDRS as ED 030 018 for $1 for microfiche and $11.70 for hard copy.

Document 11-B in this kit contains some implications for practitioners as drawn by Drs. Suydam and Riedesel from the research, and No. 11-C is a list of some current research documents on elementary mathematics entered into the ERIC system and available from EDRS.

The material in this kit was prepared pursuant to a contract with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their judgment in professional and technical matters. Points of view or opinions do not, therefore, necessarily represent official Office of Education policy or position.

IPRIAL

ANSWERS FROM RESEARCH ON---

ELEMENTARY MATHEMATICS

No. 11-A

The questions and answers in this section have been categorized under the following topics for quick reference by the reader:

TOPIC	PAGE
CONCEPT DEVELOPMENT AND LEARNING	
Achievement	3
Effect of Parental Mathematical Knowledge	3
Estimation	4
Generalization	4
Measurement	5
Motivation	6
Quantitative Concepts in Other Subject Areas	7
Readiness	7
Reasoning and Mathematics Learning	7
Reinforcement	8
Relation of Age to Achievement	9
Relation of Reading Ability to Mathematics	9
Remediation	10
Retention	10
Transfer	12
MATERIALS	
Audiovisual Devices	13
Manipulative Devices	13
Textbooks, Teachers' Manuals and Workbooks	14
MATHEMATICAL AREAS	
Counting	17
Numerals--Writing and Reading	17
Algebra	18
Geometry	18
Logic	19
Percentage	19
Probability and Statistics	19
Problem Solving	20
Ratio and Proportion	22
Sets	22

TOPIC	PAGE
MATHEMATICAL OPERATIONS	
Addition	23
Subtraction	24
Multiplication	26
Division	27
Decimals	29
Fractions	29
Mental Computation	31
Number Properties and Relations	31
Drill and Practice	32
ORGANIZATION FOR INSTRUCTION	
Acceleration and Enrichment	33
Content by Grade Level	34
Grouping Procedures	34
Time Allotment	35
Planning and Organizing for Teaching	36
Methods of Instruction	39
STUDENTS AND TEACHERS	
Effect of Teacher Background on Student Achievement	40
Inservice Training	40
Physical, Psychological, and/or Social Characteristics of Students	41
Pre-Service Training	42
Sex Differences	43
Socioeconomic Differences	44
Student Attitude and Classroom Climate	44
Student Use of Arithmetic	46
Teacher Attitudes	46
Teacher Competency	47
TEACHING METHODS AND STRATEGIES	
Checking	48
Diagnosis	48
Homework	50
Programed Instruction	50
Testing	52
MISCELLANEOUS	
Foreign Comparisons	53
Numeration Systems (Ours and Others)	54

CONCEPT DEVELOPMENT AND LEARNING

ACHIEVEMENT

Most of the studies in this category are appropriate only to the time in which they were done; the findings are not necessarily generalizable to today.

How does "modern mathematics" achievement compare with "traditional mathematics" achievement?

School Mathematics Study Group (SMSG) pupils scored higher than traditional pupils in the junior high (Cassel and Jerman, 1963), though lower scores in the eighth grade were found by Williams and Shuff (1963). For other modern programs Ruddell (1962) reported higher achievement than for traditional programs, and Payne (1965) summarized studies to conclude that modern programs are as effective as traditional programs in developing traditional mathematical skills.

EFFECT OF PARENTAL MATHEMATICAL KNOWLEDGE

What effect does the mathematical knowledge of parents have on the mathematical knowledge of children?

A very important factor in a child's learning of mathematics may be the assistance he receives from his parents. To some extent, the type and amount of assistance will be related to the parents' knowledge of the subject. Three studies found that increased parent knowledge of mathematics or classroom activities resulted in higher achievement by pupils. Duncan (1964) reported that knowledge of SMSG mathematics or classroom activities seemed to increase achievement. Mayes (DA* 1966) found that parent participation in a program resulted in higher pupil achievement. One factor that could have influenced the results of these studies is that the supportive interest of parents in their children may have been reflected or related to their willingness to gain knowledge. Another interesting dimension is added by a study by Stendler (1951). He found that generally, for preschool children, lower-social level parents emphasized counting and higher-social level parents emphasized language as a skill needed for school.

---

* Reference from *Dissertation Abstracts*. This symbol occurs throughout document.

ESTIMATION

Does teaching pupils to estimate improve achievement?

Dickey (1934) found that there was no difference in the achievement of groups who practiced estimation and those who didn't; while, in a better controlled study, Nelson (DA 1967) found that estimation was effective in increasing understanding of concepts. Faulk (1962) analyzed the estimates pupils made for a problem and found that only half gave acceptable responses. Corle (1963) found that fifth and sixth graders could estimate nearly as accurately as teachers and college students.

What are effective ways to teach estimation?

An analysis of the techniques for estimation presented in textbooks by Faulk (1962) revealed that finding sensible answers; estimating in computation to find sums, differences, products, quotients, measures, and averages; estimating answers to verbal problems and rounding off numbers were all presented, but no one text treated all of these.

GENERALIZATION

What is the relationship between generalization and mathematical achievement?

Research involving generalization is scattered widely within the field of mathematical achievement. Collier (1922) studied generalization of solutions to problems involving multiplication of fractions by whole numbers. Research by Mitchell (1929) and Henderson (1967) support the general finding that given a specific task, involving specific numbers and relationships, a student can find the solution and generalize to the broader mathematical concept. Ebert (1946) found large variation in generalization ability, depending on the mathematical concept, the student's mental age or intelligence, and the visual pattern presented. Shepard (1956) also found visual patterns and geometric shapes significant in learning mathematical concepts. Overman (1930) reports that, in a study of transfer, generalization produced over 20 percent of the transfer, more than any other means.

MEASUREMENT

What should be the grade placement of concepts of measurement?

Many studies present the levels at which various time concepts are attained: Ames (1946), Friedman (1944), Harrison (1943), MacLatchy (1951), Spayde (1953), Springer (1951, 1952). Anselmo (DA 1967) reported positive relationships between time concept scores and IQ, MA, and CA, but not SES, while Tom (DA 1967) found IQ was not so important. Washburne (1939) reported data for the Committee of Seven on linear and square measures and time. Estimation of time was also of concern to Gilliland and Humphreys (1943) and Goldstone, Boardman, and Lhamon (1958). Dutton (1967) was one of the few who experimented with teaching time concepts; he concluded that time concepts must be specifically taught to culturally disadvantaged children. In another experiment, Scott (1966) found that measurement terms in problems are not too difficult for intermediate graders. Eroh (DA 1967) found a structured program was better.

In other studies, size estimation was found to be affected by the value children gave objects (Blum, 1957), and by rewards (Lambert, Solomon, and Watson, 1949). Very little change in size constancy occurred from ages 5 to 12 (Cohen, Hershkowitz, and Chodack, 1958; Long, 1941). The greatest discrepancy between measurements and estimations was found to occur in weights and the smallest discrepancy in temperatures, with boys found to be more accurate than girls (Corle, 1960).

Paige and Jennings (1967) noted the inconsistencies of measurement content between first- and second-grade textbook series; greater agreement is found after grade three.

What materials are most effective for teaching measurement?

Programed instruction was found to be effective in teaching area concepts (Keisler, 1959), but no different from traditional instruction for teaching latitude and longitude (Spagnoli, 1965). Students using SMSG materials achieved superior growth on measurement concepts, according to Friebel (1967).

MOTIVATION

How best can motivation in learning mathematics be increased?

There are many theories about motivation and its effect on learning. Research is neither conclusive nor in agreement as to which theory is the most effective. Studies by O'Brien (1928), Brown (1932), Bouchard (1951), and Leibowitz (1966) report that knowledge of results and knowledge of competition are the most effective means to motivation. Brown reported from the junior high level and Leibowitz from kindergarten that in controlled experiments the pupil's knowledge of his own as well as his classmates' progress results in greater achievement.

What materials can be used to motivate elementary school mathematics students?

Throughout the literature there are numerous reports about various devices and games that have been used to increase student interest and hopefully achievement in elementary school mathematics. Scaramuzzi (1965) used money and its manipulation to teach arithmetic. Wilson (1922) presented work problems in the form of drama. Worden (1931) found games to be a better motivator of arithmetic accuracy than praise-punishment. Steinway (1918) found number games effective in the first grade. Richardson (1920) reported that setting definite goals or "Campaign Programs" increased achievement through motivation in grades four to eight. Reavis (1917) found that learning about classroom stocks and bonds motivated mathematics achievement. Goforth (1938) effectively used the game "ADD-O" to motivate greater mathematical achievement. It is obvious from all of these reports that, where teachers involve their students in games or imaginative programs, the mathematical achievement of the students increases.

Is individual instruction useful in motivating mathematical achievement?

With the advent of individualized instructional media there have been several studies dealing with the motivational aspects of individualized instruction. MacLatchy (1942) reported that individualized instruction in grades three and four increased the students' motivation to achieve in elementary school mathematics. As long ago as 1915 individualized instruction has been recognized as one method to increase attitude and achievement. Anthony (1915) reports increased attention and "proper" attitude when students were given individualized instruction. The limiting factor, of course, is teacher time.

What verbal technique can teachers use to increase motivation?

Hurlock (1925) reported that praise and reproof (verbal punishment) were both able to produce an increase in motivation to achieve in elementary school mathematics, as opposed to being ignored. Worden (1931) found reproof to be more motivating than praise. However, Kapos (1957) found that praise in varying amounts and in varying patterns produced excellent motivation. Hollander (DA 1968) cited evidence that verbal praise and a candy reward were more effective than no incentive or reproof.

QUANTITATIVE CONCEPTS IN OTHER SUBJECT AREAS

What effects do quantitative concepts have upon other subject areas?

The most frequently used concepts of mathematics used in other subject areas are time, measurement, money, and distance. These concepts not only permeate the curriculum of other subject areas but also the environment of every pupil. It would not be desirable or even possible to confine such topics to a mathematics text or class. However, many pupils are penalized in English or social studies for not understanding the quantitative concepts that are included in those subject areas. Jarolimek and Foster (1959) found as many as 400 quantitative concepts on a 10-page sample of one social studies text. Lyda and Robinson (1964) classified 900 concepts that were found in three social studies texts. Older studies by Partridge (1926) and Woody (1932) found similar concepts in English texts. After discovering the extent of the material contained in these sources, the researchers attempted to measure the pupils' understanding of those concepts that were found. They found that only 50 percent of the mathematical concepts found in English and social studies texts were understood by pupils using those texts. All of the researchers agree that greater emphasis should be placed upon understanding of basic quantitative concepts taught in elementary school mathematics.

READINESS

What has been ascertained about readiness?

Brownell (1938, 1951) cited evidence of how children achieve to support his contention that children are ready to begin formal arithmetic instruction in grade one. He recommends that abstract arithmetic should be translated into concrete experiences. In 1960, after comparing British and American schools, he added that children could learn more in the lower grades than we now ask. Dutton (1963) noted that 31 percent of the kindergarten children he tested were above the norm necessary for beginning systematic instruction in arithmetic. Koenker (1948) found that kindergarten pupils who had a readiness program achieved significantly higher gains on a readiness test than pupils who had a regular program.

REASONING AND MATHEMATICS LEARNING

How are process and reasoning affected by rote learning in contrast to learning by discovery?

Wilson (1967) compared learning by rote and learning by discovery. He found the discovery method superior. Meconi (1967) qualified the result by showing that pupils with high ability were able to learn under any teaching method. Previously, Brownell (1943), after extensive investigation, concluded that drill does not lead to understanding. Wohlwill (1963) supported this finding and reported that, in elementary school mathematics, understanding was achieved through relationships rather than memorized absolute rules. Earlier studies by Meyers (1928) and Rosse (1930) compared various forms of rote learning. Though not stated explicitly, both found achievement to be greater in situations that involved less absolute rote learning.

Is there a relationship between reasoning and chronological development?

Perreault (1957) discovered that the child's ability to count, to group, and to subitize proceeded in order, appearing as developmental stages. This led to the conclusion that reasoning in elementary school mathematics is related to developmental stages of the pupil. Brownell (1944), after extensive investigation, concluded that grade four is the earliest grade demonstrating maximum learning. Potter (1968) reported that among preschool children the ability to count was related to age more than any other factor. Harrison (1934) reported similarly that the ability to deal with the concept of time was also correlated with age and grade development. Beilin and Gillman (1967) reported in an excellent study the relationship between developmental stages and the language factor involved in numerical patterns. This study has major theoretical implications rather than practical applications.

REINFORCEMENT

How effective is reinforcement for increasing the learning of mathematics?

The use of reinforcement in the learning situation is an accepted teaching technique. The methods or kinds of reinforcement and the time of reinforcement can be varied in a multitude of ways. A group of well done experiments support the idea that reinforcement can and does increase learning and gives clues to the classroom teacher as to how and when to use reinforcement (Bouchard, 1951; Brown, 1932; Doherty and Wunderlich, 1968; Paige, 1966). A related study, done by Feigenbaum and Sulkin (1964), found the reduction of irrelevant stimuli more successful than reinforcement. It would seem that, by using both reinforcement and reduction of irrelevant stimuli, learning could be increased.

What type of reinforcement seems more effective and when should it be used?

One apparent and feasible way of using reinforcement to improve learning is supported by three reputable studies. By giving information on the results of tests, Bouchard (1951), Brown (1932), and Paige (1966) all found significant gains in achievement. Brown also found that boys appeared to be more easily influenced by this type of reinforcement than girls. Varying the amount of reinforcement, rather than using a constant amount, was found to be more effective for having young children change their estimations of size (Tajfel and Winter, 1963).

Most teachers have at times prompted students by giving the correct answer rather than waiting for the student to respond. McNeil (1965) found that waiting until the student had overtly responded before giving the correct answer as reinforcement significantly increased achievement, was even more effective in grade three than grade five, and seemed to be especially effective with low mental ability children. Doherty and Wunderlich (1968) found that increasing the amount of secondary reinforcement (an object or symbol that in itself has no immediate value, but has been paired with a primary reinforcer that does have immediate value) aided in increasing the number of problem-solving tasks performed by seventh- and eight-grade boys.

RELATION OF AGE TO ACHIEVEMENT

How does age and grade placement affect achievement?

The chronological age of a child may deter or facilitate his academic achievement, and the relationship should not be overlooked in evaluating achievement progress. Though the usual procedure is to assign children to grade level by chronological age, the children in a specific grade may still represent a wide range in age.

A study by Carroll (1963) found overage third-grade children scored significantly higher in arithmetic achievement, and were rated higher on attention span, independence, and social maturity when compared to underage children. The findings confirm an earlier study done by Carter (1956) which found that older children (grade one through six) seemed to have an advantage over younger children in achievement. Klausmeier and others (1958) found five physical measures of organismic age contributed very little to mental, reading, language, and arithmetic scores.

Several studies by Holmes and Finley (1955, 1956, 1957) dealing with fifth, sixth, seventh, and eighth graders found low correlations between arithmetic achievement and grade placement deviation. Grade placement deviation was determined by the difference between children's actual grade placement and the grade they would have been placed in as defined by age.

It would appear that the effect of age on achievement may diminish as age increases. Messler (1961) found no differences in achievement for eighth and ninth graders having duplicate algebra courses, and concluded that age was not detrimental to achievement.

RELATION OF READING ABILITY TO MATHEMATICS

What is the relationship between vocabulary and learning mathematics?

The ability of children to understand vocabulary or technical concepts varies greatly for individuals and generally increases with intelligence, achievement, age, and grade (Brotherton, 1948; Chase, 1961; Cruickshank, 1946). When specific training in mathematics vocabulary is carried on, Dresher (1934) and Johnson (1944) found definite gain in vocabulary and ability to solve problems. Lessenger (1925) found general reading instruction improved problem solving. Both Hanson (1944) and Treacy (1944) found a close relationship between composite reading skills and problem-solving ability. It would appear that reading ability of students, reading level of materials, and vocabulary of both must be considered as being closely interrelated with learning to solve verbal problems.

REMEDIATION

What are the causes of low achievement in mathematics?

Bernstein (1956) indicated that both intellectual and emotional factors are relevant. Easterday (1964) identified (1) low ability, (2) psychological problems which prohibit a child from functioning at his level of ability, (3) insufficient motivation, (4) inability to read and comprehend written materials, and (5) general discipline problems.

What procedures are effective with the pupils with problems in mathematics?

That planned remedial instruction improves achievement has been shown by many studies: Bemis and Trow (1942), Bernstein (1956b), Callahan (1962), Cooke (1931, 1932), Fogler (1953), Guiler (1929, 1936), Guiler and Edwards (1943) Tilton (1947). Such programs appeared to be especially effective when instruction was individualized to meet specific, diagnosed needs. Lerch and Kelly (1966) reported that a seventh-grade program planned for slow learners, with intensive teacher-pupil interaction, was successful.

Higgins and Rusch (1965) found that a programed text and a workbook were equally useful for remedial teaching. SMSG materials were successfully used with slow learners, according to Easterday (1964)

RETENTION

What instructional techniques can a teacher use to produce greater retention?

Various techniques that can be used to increase retention are suggested by research, and they generally support accepted aspects of learning theory. Gagne and Bassler (1963) found that smaller variation in task examples resulted in significantly lower retention of subordinate knowledge of elementary non-metric geometry tasks, but not of the final task. Two studies that were concerned with the retention by children of low, average, and high intelligence were by Klausmeier and Check (1961) and Klausmeier and Feldhusen (1959). Both concluded that, by assigning learning tasks appropriate for the achievement and intelligence level of a pupil, equal retention results for all pupils. Wittrock and Kessler (1965) found that giving specific and class cues in instruction are more effective than general cues for retention of previously learned concepts.

What is the relationship between "meaningfulness" and retention?

A generally accepted fact is that when something being learned has meaning to the learner and is understood by the learner, the learner will be more likely to remember or retain the learning. Several studies have investigated and compared retention resulting from meaningful learning versus mechanical learning. The findings show that teaching for meaning and understanding aid retention. A study in 1949 by Brownell and Moser found this to be true as did one by Gray (1965). Shuster and Pigge (1965) found that pupils who spent 75

to 50 percent of class time on developmental meaningful activities and less time on drill had significantly better retention than pupils who spent 25 percent of their time on developmental and meaningful activities, with proportionately more time on drill. Krich (1964) also found a meaningful method of teaching division of fractions aided retention.

With retesting, retention was found to increase in two studies by Davis and Rood (1947) and by DeWeerdt (1927). Burns (1960) concluded that intensive review as an instructional technique favors retention. The resulting retention from different methods of teaching a specific procedure was investigated in two studies. Treadway and Hollister (1963) found that teaching three cases of percentage as parts of the whole aided the average IQ pupils. Stephens and Dutton (1960) did not find any significant difference in retention when pupils taught division of fractions by the inversion method were compared with ones taught by a common denominator method.

What is the relationship between "discovery" type teaching and retention?

If either immediate recall or retention at a later date take precedence, different teaching methods may be appropriate. The intellectual characteristics of the pupils may also need to be considered in determining what type of instructional techniques to use. Worthen (1968) found that expository instruction resulted in higher immediate recall, but guided discovery favored retention. Meconi (1967), using programed material, found no differences in retention for mathematically gifted pupils with different instructional techniques. These techniques included rule and example, guided discovery, and discovery.

What can teachers do to increase pupils' retention of learning during the summer session?

Teachers are concerned about the lack of retention which is apparent after a summer vacation. The amount of loss of skill and achievement that occurs during summer months seems to vary with the child's ability, age, activities, and conditions of actual learning, especially when the first learning was done just prior to vacation. An older study by Osburn (1931) concluded that the greatest summer loss occurred in grades where subject-matter had been taught for the first time. Significant loss in computation and problem-solving scores of fifth-grade pupils seemed to be a result of use and possibly meaningful first learning in a study by Sister Josephina (1959). Scott (1967) found no systematic relationship of summer loss to the type of program, whether traditional or modern.

Two studies give teachers indications of how to decrease the amount of loss, or improve retention over the summer vacation. Dougherty (1962) found that helping children diagnose their own errors during instruction seemed to result in higher retention. Cook (1942) found that using practice materials during the summer increased retention of fundamentals for primary grade children, and the increase in retention was in direct ratio with an increase in number of weeks the practice materials were used.

TRANSFER

What kind of teaching techniques improve transfer?

The basic idea of transfer infers that something learned in one situation can be applied or used in another situation. A major concern of teachers is that pupils transfer learning from one situation to another. Two studies done in 1930 (Overman, 1930; Woody, 1930) found that emphasizing generalizations during instruction increased the amount of transfer to untaught arithmetic problems. Related to this are the results of a study by Cluley (1932) where pupils taught objectively (involving generalizations) appeared to transfer more learning than pupils who were given extra practice and/or taught by formal rules. Teaching by formal rules infers mechanical or rote instruction rather than meaningful instruction. Brownell (1949) found meaningful instruction aided transfer of learning when compared with mechanical instruction. Discovery-type instruction seems to increase transfer. Two studies (Scandura, 1964; Worthen, 1968) did find greater transfer resulted from discovery-type instruction than from expository instruction.

How can pupil ability to transfer be increased?

The transfer of learning to new concepts and situations cannot be taken for granted by teachers. Wittrock and Keisler (1965) found that specific and class cues were more effective than general cues for tranfer to new situations of previously learned concepts; but transfer to new concepts was not significantly affected by specific, class, or general cues. In an experiment by Kolb (1967) mathematical instruction was specially geared for transfer to science and transfer did occur. The instructional sequence in mathematics was constructed on the basis of a mathematical hierarchy and related to quantitative science behaviors. The use of a concept name by preschool children was related to increased transfer differentiation in a study by Spiker and Terrell (1955). It would seem that, for transfer to new concepts to occur, teachers must plan and initiate the transfer.

How much transfer of computational facts can a teacher expect?

The amount of transfer is greatest when the problems are of the same structure and transfer is to a different example of the same concept, rather than a different concept. Some older studies concerned with computational transfer found that pupils did not need to be instructed in all combinations of an operation. Knight and Setzafandt (1924) found pupils instructed in a limited set of denominators scored as well as pupils instructed in the complete set, and Olander (1931) had the same results with instruction of addition and subtraction combinations. Grossnickle (1936) found that multiplication knowledge did not transfer completely to long division, with increased errors of multiplication occurring in long division computation. It seems that a teacher can expect greater transfer of computation with similar problems and decreasing transfer with increasing differences in the types of problems, and should plan instruction that will insure transfer to different types of problems.

MATERIALS

AUDIOVISUAL DEVICES

With what topics do audiovisual devices aid in teaching mathematics?

Suppes, Jerman, and Groen (1966) reported that practice on arithmetic facts can be presented via a computer-connected teletype. Anderson (1957) reported that use of a kit of visual-tactual devices was helpful in a unit on area and volume. Howard (1950) noted that retention for a group using audiovisual aids for fractions was better.

Many other studies used audiovisual devices, but did not explicitly test their effect.

How effective is television in teaching mathematics?

Television instruction did not seem better than conventional instruction, reported Jacobs and Bollenbacher (1960). It seemed more effective, however, when seventh graders were grouped homogeneously (Jacobs, Bollenbacher, and Keiffer, 1961). Kaprelian (1961) reported a more favorable attitude toward arithmetic by fourth graders as a result of a televised course. The "Patterns in Arithemetic" television course was noted by Weaver (1965) to be as effective as a traditional course.

MANIPULATIVE DEVICES

Does the use of desk calculators, games, etc., improve learning?

Betts (1937); Fehr, McMeen, and Sobel (1956); and Triggs (1966) reported that use of a calculator for work with fundamental operations resulted in increased achievement scores. An abacus helped to produce better computation scores more than workbooks did (Earhart, 1964), while Jamison (1964) found no differences resulting from use of a large abacus, individual abaci, or the chalkboard.

Dawson and Ruddell (1955) found that manipulative materials seemed to aid achievement in division. Plank (1950) noted that Montessori materials seemed helpful for remedial work. Training with Dienes' attribute blocks was compared with use of the Greater Cleveland program by Lucas (DA 1967). He found that the attribute block group were (1) better conservers, (2) better at conceptualization of addition and subtraction, (3) not as good in computation, (4) no better on problems, and (5) slightly better at multiplication.

In general, such materials seem to be more effective for slow and average learners than for those achieving above average.

Does the use of Cuisenaire materials improve mathematical achievement and understanding?

Brownell (1963), after interviewing English children who had used the Cuisenaire program, reported that they responded more quickly to simple combinations than did traditionally taught students, and used more sophisticated solutions for unknown combinations. However, the traditional group was more accurate. No differences were noted in understanding or problem-solving. In another study (Brownell, 1968), Brownell found that Scottish students using the Cuisenaire program had less instruction time and demonstrated greater maturity of thought processes than conventional groups. The Cuisenaire group did not, however, perform better in verbalizing answers. For English students, the conventional group ranked higher, with Cuisenaire and Dienese programs about equal on conceptual maturity. All three programs were similar for problem solving.

The Cuisenaire program taught traditional subject matter as well as the traditional method when measured by an achievement and a traditional test, according to Hollis (1965). Additional concepts and skills were acquired by the Cuisenaire pupils. Nasca (1966) added evidence to support this. Lucow (1963, 1964) reported that the Cuisenaire program was as effective for third graders as the traditional program in teaching multiplication and division. On the other hand, Passy (1963, 1964) found that third-grade children using Cuisenaire materials achieved significantly less than other groups.

Fedon (DA 1967) noted that maximum manipulation was the essential factor, and first graders using Cuisenaire materials achieved slightly less than those using an eclectic approach. Callahan and Jacobson (1967) found that the rods could be used effectively with retarded children.

TEXTBOOKS, TEACHERS' MANUALS AND WORKBOOKS

How have textbooks changed over the years?

An extensive analysis of 59 arithmetic textbooks for 150 years of publication (1790 to 1940) was done by Smith and others (1942, 1943, 1945). Basic changes that occurred in textbooks over the years were the inclusion of inductive method, increased "real life" emphasis, increased importance of learner interest, and change in content from emphasis on subject matter to meeting needs of user. In an analysis of teacher texts and student series, Hicks (1968) found a wide diversity of topics with less agreement on relevant topics for teacher texts than for pupil texts. Dooley (1960, 1961) studied the relationship of research to content on 12 topics, finding that clear, concise, exact recommendations were incorporated into textbooks within 5 years. Others used textbook analysis to ascertain the amount of content for specific topics.

How effective are modern mathematic textbooks?

The impact of SMSG materials on seventh-, eighth-, and ninth-grade achievement has been investigated and reported in the literature. A good study by Williams and Shuff (1963) compared programs using SMSG and traditional materials. They found that the seventh- and ninth-grade groups did not differ significantly in achievement gain. The only group that made any significant achievement gain was the eighth-grade traditional group. Contradictory findings were reported in a study by Cassel and Jerman (1963) in that pupils of the same grade levels receiving SMSG instruction had statistically significant achievement when compared to students who had traditional instruction. Friebel (1967) found that the SMSG group achieved significantly more in arithmetic reasoning and on measurement concepts.

A related study by Nelson (1965) investigated the achievement of high-ability pupils who used high or low level SMSG textbooks. Generally, there were no significant differences in terms of the textbook used, but the high-ability, low-achieving students tended to perform better when using the lower level SMSG materials.

Hungerman (1967) found that groups taught with SMSG materials in grades four, five and six scored better on contemporary tests, while traditional groups scored better on traditional tests.

Hughes (DA 1968) found that SMSG materials had had a greater impact on post-1960 commercially published textbooks than other materials had had.

How do teachers use textbooks and teacher's manuals?

Folsom (1960) found that about half of the teachers she studied did not use the manual, but had all pupils use the textbook. Little use of the concrete and semiconcrete materials suggested by the manuals was made. Teachers particulary liked the combined textbook-manual. Butt (DA 1967) suggested a list of criteria for writing and producing textbooks.

What is the reading level of current mathematics textbooks?

Research indicates that many problem-solving difficulties are actually reading difficulties. The assumption that a text for a certain grade is based on the reading level of that grade may be a false assumption. Buswell (1931) indicated this was a problem of concern many years ago, and recent research indicates the problem is still with us. Smith and Heddens (1964) found the reading level of experimental mathematics materials was usually above the grade level of use. They also found the same true of five commercial textbooks, with great variation between and within the textbooks. A study by Repp (1960) which may be relevant found 1,379 or more new words introduced in third-grade textbooks. It seems realistic to investigate the reading level and increase in new vocabulary when selecting textbooks, and not to make the assumption the text will be appropriate for the grade level. Covington (DA 1967) reported that the reading level of a series of modern texts was too difficult for third and fourth graders. Reed (DA 1960) found little agreement between vocabularies in reading and arithmetic texts.

What can be said about the specific vocabulary (technical language) used in textbooks?

The frequency of specific vocabulary in textbooks has been investigated by many researchers in the past. Of words occurring five or more times, Brooks (1926) found 237 and Gunderson (1936) found 252. A wide variation in the actual vocabulary or technical terms is found between textbooks (Pressey and Elean, 1932; Repp, 1960; Willey, 1942). Currey (DA 1966) reported that new terminology is confusing to low-socioeconomic-level first graders. Stevens (DA 1966) found that, between 1956 and 1964, the vocabulary load increased more than 40 percent.

Are workbooks effective in increasing mathematical achievement in elementary school?

Durr (1958), in an extensive study of workbooks in grades four to eight, found workbooks to be an effective aid in mathematical achievement in grades four and five. There were no significant differences attributable to workbooks found in grades six and above. Andreen (1938) found wide variations in achievement, depending on the use that teachers made of workbooks. When teachers relied on the workbooks to do their teaching for them, very little gain in achievement was noted. Stutler (1962) found that examining pupils' workbooks was a measure of mathematical achievement. In general, research indicates that, where proper use of workbooks is practiced, mathematical achievement can be increased.

MATHEMATICAL AREAS

COUNTING

When should instruction in counting begin?

Woody (1931) noted that children had a considerable knowledge of counting before formal instruction began (at grade 2 for his groups). This has been supported by studies with preschool, kindergarten, and first-grade children. Woody also reported that only 2 percent of the parents indicated that they did not teach their children to count.

Most low-intelligence fourth graders could count by 2's; those in the average group could count by 3's to 16's; and in the high group, children could count by 3's to 23's (Feldhusen and Klausmeier, 1959).

How much emphasis should be placed on sets before beginning to teach counting?

Studying kindergarten children Carper (1942) reported that the amount of grouping decreased and counting increased as pictorial context was increased. Dawson (1953) found that the greater the complexity and size of groups, the more counting occurred. Children aged 8 to 10 could only grasp a set of four objects, while those aged 10 to 12 could grasp five (Freeman, 1912). Various grouping patterns were studied by Brownell (1928), who reported that recognition of groups of dots was related to the size of the group but no numbers from 3 to 12 were more difficult. Children counted at first, then proceeded to more mature methods.

What techniques are most effective for teaching counting?

Dawson (1953) reported that geometric presentations might precede pictorial forms.

NUMERALS--WRITING AND READING

How effective is the present teaching of numeral writing?

Little research has been done to answer this question in recent years. Hildreth (1932) found that the numerals 4, 8, and 2 were most difficult for kindergarten and first-grade children to write, while 3, 9, and 7 were easiest. Newland (1930) reported that, for third through ninth graders, the order of illegibility was 5, 7, 2, 0, 4, 9, 8, 6, 3, 1. Two implications from most studies which are inherently sensible are that numeral writing must be taught or retaught at each grade level, and the need for legibility must be stressed throughout life. Buchanan (DA 1967) found that kindergarten pupils were able to learn to write numerals legibly, but this did not facilitate arithmetic conceptualization.

How can writing and reading numerals be effectively taught?

Most of the research which attempts to answer this question is found in the literature on the teaching of reading, since many of the same principles apply. Wheeler and Wheeler (1940) reported some success with the use of a game to teach children to read numerals, but this was under a rote teaching philosophy. The reading and writing of numerals is today connected more closely with the study for understanding of the decimal system.

ALGEBRA

What is the effect of teaching algebra?

Braverman (1933) noted that algebra instruction resulted in increased arithmetic scores. Cassell (1963), reporting on the effect of SMSG instruction, noted increased scores in both arithmetic and algebra. No significant differences between programed or traditional materials on equations and inequalities were found by Kalin (1962), and Messler (1961) found no significant differences after an algebra course. However, Banghart, McLaulin, Wesson and Pikaart (1963) found that, on a comparison of a traditonal program and a modern mathematics program which included algebra, the modern program resulted in higher achievement scores.

GEOMETRY

What geometry can be effectively taught in the grades?

D'Augustine (1964) identified the following as highly teachable via programed instruction: interior, exterior, and boundary points; congruency; simple closed curves; triangle properties and definitions; collinearity; finite and infinite points; and properties of lines and line segments. Weaver (1966) reported on an inventory for geometric understanding; he found no significant differences between conventional and modern classes. Instruction in coordinate geometry was reported effective by Herbst (DA 1968) at fifth-grade level, and St. Clair (DA 1968) taught symmetry.

How can the vocabulary of geometry be most effectively developed?

Shepard and Schaeffer (1956) noted the knowledge of the name of an object helped pupils to achieve on a discrimination task.

What is the best sequencing of geometric topics?

Gagne and Bassler (1963), in connection with building a hierarchy, found that the group having the smallest variety of task examples in nonmetric geometric materials retained less.

LOGIC

What materials are most effective in teaching ideas of logic?

The WFF'N Proof game aided logic scores (Allen, 1965), as did the SMSG program (Scott, 1965) and a program by Suppes (1964 ; Suppes and Binford, 1965).

PERCENTAGE

When should percentage be introduced?

Kenney and Stockton (1958) found that the three upper intelligence level groups made significant progress in learning about percentage in grade seven; Kircher (1926) reported that only about one quarter of all pupils tested at grade eight had acquired "an intelligent understanding." McCarty (DA 1966) reported success in teaching percentage at grades four, five and six.

How should "cases" of percent be taught--related or unrelated?

Guiler (1964) reported difficulty levels at the ninth grade as follows: finding a percent of a number, 51.6 percent; finding what percent one number is of another, 47.7 percent; finding a number when a percent of it is known, 94.0 percent; finding the result of a percent increase or decrease, 72.2 percent; and finding a percent of increase or decrease, 88.2 percent. Tredway and Hollister (1963) reported that teaching the three cases of percentage as related parts of a whole process provided for better retention.

What method should be used in teaching percent--ratio, unitary analysis, equations, formulas, decimals?

Wynn (DA 1966) found no significant differences in achievement or retention between unitary analysis, formula, or decimal methods.

Can percent be effectively taught in the context of science and social studies?

Reavis (1957) found a project on stocks and bonds was effective; and Riedesel (1957) noted that most textbooks then currently in use had one to four pages on discounting of bank loans.

PROBABILITY AND STATISTICS

What ideas concerning measures of central tendency can be developed?

Burns (1963) found that understanding of the mode and the mean could be taught in grade four, while the median was a more difficult concept.

What concepts of probability can be effectively taught?

Probability learning was found to occur from the environment and was maximized by rewards (Messick and Solley, 1957). Smith (DA 1966) reported that topics in probability and statistics could be taught to most seventh graders. Ojemann, Maxey, and Snider (1965) found that third graders learned to make predictions when proportions were known, seeking more information before making predictions.

PROBLEM SOLVING

The solving of verbal or word problems has long been one of the areas of elementary school mathematics that has concerned teachers and created anxiety in children. Problem solving has always been a favorite topic of persons doing research on elementary school mathematics instruction. In fact, there are probably more practical answers from research to help in the improvement of children's problem solving skills than for any other areas of the elementary school mathematics curriculum.

How do pupils think in problem solving?

Studies by Stevenson (1925) and Corle (1958) revealed that pupils often give little attention to the actual problems; instead, they almost randomly manipulate numbers. The use of techniques such as "problems without number" can often prevent such random attempts.

What are the characteristics of good problem solvers? of poor problem solvers?

Researchers have identified a number of factors that are associated with high achievement in problem solving. Conversely, the lack of those factors is associated with poor problem solvers. Some of these traits are intelligence, computational ability, ability to estimate answers, ability to use quantitative relative relationships that are social in nature, ability to note irrelevant detail, and knowledge of arithmetical concepts. (See Engelhart, 1932; Stevens, 1932; Alexander, 1960; Hansen, 1944; Cruickshank, 1948; Chase, 1960; Beldon, 1960; Laughlin, 1960; Kliebhan, 1955; Butler, 1955; Klausmeier and Laughlin, 1961; Balow, 1964; Babcock, 1954.)

What is the importance of the problem setting?

Researchers such as Bowman (1929, 1932), Brownell (1931), Hensell (1956), Evans (1940), Sutherland (1941), Wheat (1929), and Lyda and Church (1964) have explored the problem setting. Findings are mixed, with some researchers suggesting true-to-life settings while others suggest more imaginative settings. While the evidence appears to be unclear, one thing does emerge: problems of interest to pupils promote greater achievement in problem solving. With today's rapidly changing world it seems unreasonable that verbal problems used in elementary school mathematics could sample all of the situations that will be important to pupils now and in adult life. Perhaps the best suggestion for developing problem settings is to take situations that are relevant for the child. Thus, a problem on space travel may be more "real" to a sixth grader than a problem based upon the school lunch program.

How does the order of the presentation of the process and numerical data affect the difficulty of multistep problems?

Burns and Yonally (1964) found that pupils made significantly higher scores on the test portions in which the numerical data were in proper solution order. Berglund-Gray and Young (1932) found that, when the direction operations (addition and multiplication) were used first in multistep problems, the problems were easier than when inverse operations (subtraction and division) were used first. Thus, an "add-then-subtract" problem was easier than a "subtract-then-add" problem.

What is the effect of vocabulary and reading on problem solving?

Direct teaching of reading skills and vocabulary directly related to problem solving improves achievement (Robertson, 1931; Dresher, 1934; Johnson, 1944; Treacy, 1944; VanderLinde, 1964).

How does wording affect problem difficulty?

Williams and McCreight (1965) report that pupils achieve slightly better when the question is asked first in a problem. Thus, since the majority of textbook series place the question last, it is suggested that the teacher develop and use some work problems in which the question is presented first.

What is the readability of verbal problems in textbooks and in experimental materials?

Heddens and Smith (1964) and Smith and Heddens (1964) found that experimental materials were at a higher reading difficulty level than commercial textbook materials. However, they were both at a higher level of reading difficulty than that prescribed by reading formula analysis.

What is the place of understanding and problem solving?

Pace (1961) found that groups having systematic discussion concerning the meaning of problems made significant gains. Irish (1964) reports that children's problem solving ability can be improved by (1) developing the ability to generalize the meanings of the number operations and the relationships among these operations, and (2) developing an ability to formulate original statements to express these generalizations as they are attained.

Should the answers to verbal problems be labeled?

While Ullrich (1955) found that teachers prefer labeling, there are many cases in which labeling may be incorrect mathematically. For example:

Incorrect	Correct
10 apples	10
6 apples	+6
16 apples	16 apples

Does cooperative group problem solving produce better achievement than individual problem solving?

Klugman (1944) found that two children working together solved more problems correctly than pupils working individually. However, they took a greater deal of time to accomplish the problem solutions. Hudgins (1960) reported that group solutions to problems are no better than the independent solutions made by the most able member of the groups.

What is the role of formal analysis in improving problem solving?

The use of some step-by-step procedures for analyzing problems has had wide appeal in the teaching of elementary school mathematics. Evidence by Stevens (1932), Mitchell (1932), Hanna (1930), Bruch (1953), and Chase (1961) indicated that informal procedures are superior to following rigid steps such as the following: "Answer each of these questions: (1) What is given? (2) What is to be found? (3) What is to be done? (4) What is a close estimate of the answer? and (5) What is the answer to the problem?" If this analysis method is used, it is recommended that only one or two of the steps be tried with any one problem.

What techniques are helpful in improving pupils' problem solving ability?

Studies by Wilson (1922), Stevenson (1924), Washburne (1926), Thiele (1939), Luchins (1942), Bemis and Trow (1942), Hall (1942), Klausmeier (1964), and Riedesel (1964) suggest that a number of specific techniques will aid in improving pupils' problem-solving ability. These techniques include: (1) using drawings and diagrams, (2) following and discussing a model problem, (3) having pupils write their own problems and solve each others' problems, (4) using problems without number, (5) using orally presented problems, (6) emphasizing vocabulary, (7) writing mathematical sentences, (8) using problems of proper difficulty level, (9) helping pupils to correct problems, (10) praising pupil progress, and (11) sequencing problem sets from easy to hard.

Note: There are many suggestions from research concerning verbal problem solving. It is suggested that the reader check the specific sources listed for other problem solving suggestions and for the representative material presented therein.

RATIO AND PROPORTION

How early in the grades can ratio and proportion be effectively introduced?

McCarty (DA 1966) reported success in teaching ratio at grades four, five and six.

SETS

Does the teaching of the notation of sets improve pupils' understanding and ability to deal with numbers?

Smith (DA 1968) found that students who received instruction in set theory showed significant superiority in logical reasoning.

MATHEMATICAL OPERATIONS

ADDITION

What is the difficulty level of the various addition combinations?

MacLatchy (1933) found that (1) the easiest combinations were those in which 1 is added to a larger number, (2) a combination and its reverse form were not of equal difficulty, (3) adding a smaller number to a larger number was easier than the reverse form, and (4) combinations which contain a common addend were not of equal difficulty. Wheeler (1939) developed a rank order of difficulty of addition facts. It should be noted that these studies occurred before the extensive teaching of the commutative and associative properties. In programs where number properties are emphasized, the difficulty of combinations may be different than that reported above. At the present time studies using computer-assisted instruction are being conducted concerning the difficulty of basic addition and subtraction combinations. These findings should add to the pool of knowledge concerning this topic.

How can addition facts be effectively taught?

Researchers have found that (1) pupils with good counting facility learn addition facts effectively (MacLatchy, 1933); (2) teaching addition and subtraction facts together may result in higher achievement (Buckingham, 1927); (3) corrective work results in score-improvement on tests of basic facts (Wilson, 1954); (4) teaching addition combinations "indirectly" (practice within examples) rather than "directly" (in isolation) results in superior achievement (Breed and Ralston, 1936); (5) independent work improves mastery of the addition facts (Wilburn, 1945); and (6) use of simple manipulative materials increased understanding more than use of only pictures (Ekman, DA 1967).

What readiness should occur before formal addition is introduced?

MacLatchy (1932) found that pupils who were proficient in counting tended to have greater success in formal addition. Other findings point to the importance of developing counting and the ability to recognize the number of a set as good background experiences preceding addition. Also, Brownell (1928) found that thorough understanding of concrete numbers resulted in transition to abstract number with less difficulty, and that difficulty with additive combinations were results of immature methods or lack of understanding of the relationships between experience with concrete and abstract.

What procedures improve achievement in column addition?

Buckingham (1927) found that children taught to add columns downward achieved higher scores than those taught to add upwards. Ballenger (1926) found that dividing a column into two parts and adding each separately resulted in greater accuracy for pupils who could not achieve accuracy on longer columns.

How do pupils think when performing higher-decade addition?

Flournoy (1956, 1957) found that (1) the majority of children first noted the basic addition fact ending when performing higher-decade addition, recording first ones, then tens; (2) when bridging was involved, the carrying method was most frequently used; and (3) some children used different methods for horizontal and vertical forms.

SUBTRACTION

What type of subtraction situation should be used for introductory work?

In an excellent study, Gibb (1956) found that the highest degree of pupil attainment was on take-away problems and the lowest level on comparative problems. She also found that additive problems took a longer time to complete. Schell and Burns (1962) found no differences in performance on the three types of subtraction problems. However, take-away problems were considered by pupils to be easiest.

Coxford (DA 1966) found that the procedure based on removal of a set from a set with no explicit use made of the relationship between subtraction and addition led to greater immediate proficiency than the more explicit procedure. Osburne (DA 1967) reported that a set-partioning-without-removal approach resulted in greater understanding than the take-away approach.

What are effective methods of teaching subtraction facts?

Gibb (1956) found that pupil performance was better on subtraction problems in semiconcrete context than in concrete context and lowest in abstract context. This suggests that pupils should be given wide semiconcrete and concrete experiences before proceeding to learn the subtraction facts. Buckingham (1927) found that pupils learned subtraction facts slightly more easily when they used a subtractive method rather than an additive method.

How should renaming in subtraction be taught?

Over the years researchers have explored procedures for teaching renaming (borrowing) in subtraction. Four different (or partially) different methods have often been explored. They are (1) take-away-renaming (decomposition), (2) take-away-equal additions, (3) additive-renaming (decomposition), and additive-equal additions. They are explained below:

Take-away-renaming

84	80 + 4	Six cannot be subtracted from 4
- 56	50 + 6	
	70 + 14	Rename
	50 + 6	Think 14 minus 6
	20 + 8 = 28	Think 70 minus 50

Take-away-equal additions

84	80 + 4	Six cannot be subtracted from 4
-56	50 + 6	
	80 + 14	Add 10 ones to 4
	60 + 6	Add 1 ten to 50
	20 + 8 = 28	Subtract

This procedure is based on the principle that, if both terms are increased by the same amount, the difference (remainder) is unchanged. This property is referred to as compensation.

6	6 + 2	8
-3	3 + 2	-5
3		3

Additive-renaming

84	80 + 4	
-56	50 + 6	
	70 + 14	Rename
	50 + 6	Think 6 plus what number + 14?
		Think 50 plus what number = 70?

Note that renaming is done in the same manner as in the classroom situation described above. The difference is in using "additive thinking" rather than "take-away" thinking.

Additive-equal additions

84	80 + 4	Six cannot be subtracted from 4.
-56	50 + 6	
	80 + 14	Add 10 ones to 4
	60 + 6	Add 1 ten to 50
		Think 6 plus what number = 14?
		Think 60 plus what number = 80?

In a classic study--Brownell (1947)--teaching of borrowing with meaning was more effective in both the equal additions and decomposition method. He also found that rational decomposition was superior to equal additions when the criteria were understanding and transfer, while mechanical teaching using equal addition produced smoother and faster performance.

Other findings comparing methods of teaching borrowing are: (1) equal-additions procedures produced fewer errors than decomposition (Osburn, 1927); (2) the additive method resulted in greater accuracy, while decomposition was faster (Beatty, 1920); (3) few children taught the equal-addition method continued to use it (Taylor, 1919); (4) equal-additions was more accurate and faster than

decomposition (Roantree, 1924; Johnson, 1931); (5) decomposition was more accurate than equal-additions, and there was no difference in speed (Rheins and Rheins, 1955); (6) decomposition was more popular with teachers (Wilson, 1934). Overall, it is reasonably safe to say that the decomposition method develops greater understanding, while the equal-additions method is slightly faster.

How does the use of "crutches" affect teaching renaming in subtraction?

Brownell, Kuehner, and Rein (1939) and Brownell (1940) examined the method as a "crutch" to borrowing in subtraction and found a significant decline in errors when the crutch was used. All but a small percentage of the children gave up the crutch readily.

MULTIPLICATION

What procedures are effective in learning basic multiplication combinations?

Brownell and Carper (1943) found that: children taught mainly by drill did not have complete meaningful learning at the end of grade five, but did have accuracy; habituation was used more frequently with easy combinations than with difficult ones; there were no high correlations between rate and CA or achievement; a moderate relationship between IQ and accuracy existed in grades three and four; and there were higher median scores for girls than boys in lower grades. Wilson (1931) found that both bright and dull children learned equally well. Fowlkes (1927) found that a method using printed materials with a little teacher comment was efficient in teaching basic facts.

Clemmons (1928) found that specific drill reduced the error rate of pupils and zero facts proved to be difficult. Harvey and Kyte (1965) found that a program of diagnosis and remediation was effective.

At what grade level should multiplication be introduced?

Not many years ago multiplication was first introduced in grade three. The present practice is to introduce multiplication in grade two. Earlier studies by Brownell (1943,1944) indicate that children were ready for multiplication combinations in third grade and were successful in learning them, that progress in accuracy on multiplication combinations was greatest in the fourth grade, and that pupil knowledge of multiplication facts increased in the fifth grade.

Should the equal-addition or the Cartesian product approach be used for introductory work in multiplication?

In a good study Hervey (1966) found that: (1) Equal-additions multiplication problems were less difficult to solve and conceptualize, and less difficult to select a "way to think" than Cartesian product problems: (2) Cartesian product problems were more readily solved by high achievers in arithmetic than by low achievers, by boys than by girls, and by those with above average intelligence, though this was not substantiated with data.

Can pupils use the distributive property?

Gray (1965) found that:

(1) A program of arithmetic instruction which introduced multiplication by a method stressing understanding of the distributive property produced results superior to methods currently in use.

(2) Knowledge of the distributive property appeared to enable children to proceed independently in the solution of untaught multiplication combinations.

(3) Children appear not to develop an understanding of the distributive principle unless it was specifically taught.

(4) Insofar as the distributive property is an element of the structure of mathematics, the findings tend to support the assumption that teaching for an understanding of structure can produce superior results in terms of pupil growth.

Schell (1968) found that, when third-grade pupils were taught basic facts of multiplication and the distributive property, they learned to use distributive property in two lessons plus a review lesson. Distributive property items were more difficult than nondistributive property items. Pupils scoring high on nondistributive items performed well on distributive items, but low-scoring pupils had more difficulty with distributive than nondistributive property items.

Hall (DA 1967b) found that stress on the commutative property was effective with commuted combinations.

DIVISION

Is the subtractive or the distributive approach to division most effective?

During the 1940's and 1950's the distributive approach to division was typically taught in elementary school mathematics textbooks. With the beginnings of "modern mathematics" the subtractive approach became much more popular. The two approaches can be contrasted below:

Distributive

```
        20 r8
23 / 468
     46
     --
      8
```

Think: "How many 23's in 400?" etc.

Subtractive

```
23 / 468
     238   10   Use any reason-
     ---        able estimate
     238
     200   10
     ---   --
       8   20 r 8
```

Dawson and Ruddell (1955) report that the use of the subtractive concept resulted in significantly higher achievement on immediate and delayed recall tests. They also found that a greater understanding of division and its inter-relationships with other operations resulted from the study of division using the subtractive concepts and manipulative materials.

What is the role of "measurement" and "partition" division in the learning sequence?

Measurement division involves problems of the type: If each boy is to receive 3 apples, how many boys can share 12 apples?

Partition division involves problems of the type: If there are 4 boys to share 12 apples, how many will each receive?

Zweng (1964) studied measurement, partitive, and rate concepts of division, finding the partitive division problems were significantly more difficult than measurement problems. She also reported that rate problems may be easier than basic problems, and partitive problems were more difficult than both basic measurement and rate measurement problems. Scott (1963) made use of two algorithms for division, using the subtractive algorithm for measurement situations and the distributive algorithm for partitive division situations. He suggested that: (1) the use of the two algorithms neither confused nor presented undue difficulty for young children; (2) teaching children to use two algorithms demanded no more teaching time than teaching only one algorithm; and (3) children taught both algorithms had a greater understanding of the division operation than those taught by only one algorithm.

What is the most effective method of teaching pupils to estimate the quotient?

For early work in estimation of the quotient in division, two suggestions are usually made. There is the "apparent" method which suggests that the pupil look at the first digit of the divisor and the "increase-by-one" or "round-up" method in which the pupil is to increase the first digit of the divisor by one thus 32 would become 40. Grossnickle (1937) found that: (1) There were no significant differences between groups learning the apparent and the increase-by-one methods of quotient estimation, on either correct or estimation scores, and (2) there was no significant difference in the mean number of computational errors made when using the two methods.

While little research has been conducted to test the best method of estimating as far as pupil achievement is concerned, a number of studies have been conducted on the efficiency of various procedures. Morton (1947) analyzed 40,014 examples and found that (1) the increase-by-one method is correct 79 percent of the time when the divisors end in 6, 7, 8, or 9; (2) the "apparent" method is correct 72 percent of the time when divisors end in 1, 2, 3, or 4; (3) for any divisor ending in 1 to 9, the apparent method is correct 53 percent of the time, and the increase by-one method, 61 percent. Karstens recommends that the "second figure 5" divisors (25, 35, etc.) should be rounded upwards, since more correct trial divisors result.

Osburn (1950) analyzed division examples with divisors ending in 6, 7, 8, or 9, using a dichotomy, and revealed that the apparent method (Rule I) is successful in 4,800 cases where increase-by-one method (Rule II) is also successful; Rule I fails in 9,846 cases where Rule II is successful; Rule I is successful in 1,885 cases where Rule II fails; and Rule I fails in 2,099 cases where Rule II also fails. Osburn (1946) noted that the apparent method of estimating the quotient, with the instruction to try a quotient figure less by 1 when a subtrahend is too large, could enable the learner to handle all but 5 percent of any long division examples. Grossnickle (1931, 1932a, 1932b 1939, 1945, 1946) also analyzed large numbers of division examples.

What are the difficulty levels of division combinations?

Brueckner and Melbye surveyed to ascertain the difficulty levels of division. They reported that the sequence of difficulty from easy to hard is: (1) apparent quotient is true quotient (M.A. 10 to 11 years), (2) one-figure quotients, apparent quotient is not true quotient (M.A. 13 to 14 years); (3) two- and three-figure quotients, apparent quotient is not true quotient (M.A. 14 to 15 years).

Is it better to teach "long division" or "short division"?

"Long division: is the form $3\overline{)\begin{array}{r}152\\456\\3\\\hline 15\\15\\\hline 6\end{array}}$ while "short division" is the form $\frac{3/456}{152}$. Olander (1932) reports that most pupils chose to use long division. However, there was some preference for short division by good students. Grossnickle (1934) found that more errors were made by pupils using only short division. John (1930) also reports that the use of the long form was conductive to greater accuracy than was the use of the short division form.

DECIMALS

How should decimals be related to place value?

In studying methods for placing the decimal point in the quotient, Flournoy (1959) found that multiplying by power of 10 was more successful than the subtraction method.

FRACTIONS

How can operations with fractions be taught effectively?

Fincher and Fillmer (1965), Traweek (1964), Greatsinger (DA 1968), Levin (DA 1968), and Wilson (DA 1968) found operations with fractions could be taught effectively by programed instruction materials. Austin (DA 1966) reported both constructed and multiple choice formats were successful. Miller (1964) reported that written lesson plans plus automated practice machines were superior to use of the textbook plus concrete materials in teaching multiplication with fractions.

Krich (1964) reported that low IQ groups taught division with fractions meaningfully or mechanically did not differ in achievement, while the normal IQ group taught meaningfully scored higher on a retention test than a mechanically taught group.

Gunderson and Gunderson (1957) found that second graders could understand fractions when they used manipulative materials. Audiovisual aids also helped fifth and sixth graders (Howard, 1950).

What is the best method for finding the common denominator for addition with fractions?

Anderson (DA 1966) found no differences for students using classes of equivalent fractions or factoring denominators when adding with unlike, unrelated fractions. Brownell (1933) evaluated the use of multiplication by the identify element to form a common denominator before adding with fractions. Labeling it a "crutch," he found children tended to drop it when simpler procedures were found.

What is the best method for teaching division with fractions?

Capps (1962, 1963) reported that the inversion method of teaching division with fractions was better for achievement on multiplication with fractions than the common denominator method, but on a retention test the inversion group remained at the same level while the common denominator group increased in achievement. Stephens and Dutton (1960) indicated that neither method was better on a retention test. Bergen (1966) cited evidence indicating the reciprocal and inversion methods were superior to the common denominator method. Bidwell (DA 1968) reported that the inverse operation method was superior to the complex fraction and common denominator methods in both structure and computational skills.

What is the best sequence for teaching division with fractions?

Hirsch (1951) found that divisin with fractions was easiest when the division sign was used (2 3/4 $\div$ 3 1/7). Next in order was "divide 3/4 by 5," followed by "divide 8 by 2 1/3."

What errors are commonly made when children compute with fractions?

Brueckner (1928) found that errors with fractions could be attributed to (1) computation, (2) lack of comprehension of the process involved, (3) inability to reduce fractions to lowest terms, and (4) difficulty in changing improper fractions to whole or mixed numbers. Shane (1938) found errors were caused by (1) difficulty in "reduction" in addition with fractions, (2) difficulty with "borrowing" in subtraction with fractions, (3) faulty computation in multiplication with fractions, and (4) use of the wrong process in division with fractions. Romberg (1968) reported that pupils using modern textbooks failed to cancel when multiplying with fractions more often than those using conventional texts.

Scott (1962) found that fifth graders made many more errors in subtracting with fractions involving regrouping than in whole number subtraction with regrouping, since pupils tended to relate the process to the decimal scale. Hinkelman (1956) found that fifth graders knew an average of three of 10 principles of fractions, while sixth graders knew four.

Diagnosis of errors in work with addition and subtraction with fractions did not seem to aid achievement, according to Aftreth (1957, 1958). Guilder (1936) used individualized group remedial work to improve scores on tests with fractions.

MENTAL COMPUTATION

What effect does the teaching of mental computation have upon pupil achievement?

Improved ability to solve oral problems was reported by Flournoy (1954), while Petty (1965) found no significant differences between groups who did or did not use pencil and paper. Olander and Brown (1959) found that pupils had great difficulty when not allowed to use paper and pencil.

What techniques are best for improving mental computation?

A specified time allotment and step-by-step planned sequence of material seem necessary, according to Payne (DA 1966).

Wolf (1960) found that films and printed materials were equally successful as vehicles for presenting problems for mental computation.

Flournoy (1957) suggested the following experiences: (1) Learning short-cuts for each operation, (2) practice in solving for both exact and estimated answers after listening to orally presented problems, (3) constructing problems, (4) learning to use rounded numbers, (5) realizing the importance of properly interpreting quantitative terms, and (6) learning to read and use graphs and tables. She noted (Flournoy, 1959) that about 10 minutes per day should be spent on planned mental computation exercises.

Olander and Brown (1959) found that visually presented problems were easier than orally presented ones; this visual technique of presenting problems on flashcards was also used by Hall (1947) and Sister Josephina (1960). However, because pupils do not do well on something with which they have received little practice may indicate that they need more practice.

NUMBER PROPERTIES AND RELATIONS

Should inequalities come before, after, or at the same time as equalities?

Holmes (1963) reported that finding a subset with identical number properties was more difficult than matching sets with the same properties.

When should formal number properties be taught? How should they be taught?

At grade seven it was found that basic properties of addition were not clearly understood, with the distributive property apparently most difficult (Flournoy, 1964). Hinkelman (1956) found that only three of ten fraction principles were known at grade five, while four were known at grade six. Attainment of the concepts of commutativity, closure, and identity was found for pupils in grades two and four by Bauman (DA 1966), while Schmidt (DA 1966) reported instruction on the commutative, associative and distributive properties was effective at the fourth-grade level. Gravel (DA 1968) showed that certain relations could be taught at grade six.

DRILL AND PRACTICE

How much time should be devoted to drill and practice?

Hahn and Thorndike (1914) reported that periods of about 20 minutes were most effective, while Meddleton (1956) cited stronger evidence to show that systematic, short review work produces higher achievement. In a more recent well-done study, Shipp and Deer (1960) found that less than 50 percent of class time should be spent on practice activities, since increased achievement resulted when up to 75 percent of the time was spent on developmental activities. This finding has been supported by Shuster and Pigge (1965); Zahn (1966), and Hopkins (DA 1966).

What type of drill procedures are most effective?

Greene (1930) summarized studies which showed that drill must be constructed to fit a particular purpose and type of use, and this connection of drill with a purpose and the topic under study has been found to be of most help in more recent studies, too. Motivation and functional experiences are important (Harding and Bryant, 1944; Hoover, 1921; Lutes, 1926). Distributed practice is most helpful, rather than concentrated practice, according to Knight (1927). Children should use practice materials on their own difficulty level and progress at their own rate (Moench, 1962). Varying the type of drill and the use of "frames" were found to be effective by Sandefur (DA 1966).

Where in the sequence of learning mathematics is drill most effective?

After effective teaching is the time for drill, stated Brownell and Chazal (1935), and this has been generally supported and accepted.

ORGANIZATION FOR INSTRUCTION

ACCELERATION AND ENRICHMENT

Has acceleration proven to be effective for the superior pupil?

Aftreth and MacEachern (1964) found that both an acceleration and an enrichment program were effective. Townsend (1960) and Ivey (1965) offered further evidence to show that acceleration is possible, even when not limited to those with high IQ scores. Jacobs, Berry, and Leinwohl (1965) indicated that the effect of acceleration was observable only over a period of time.

Klausmeier and Ripple (1962) found no unfavorable academic, social, emotional, or physical correlates of acceleration from second to fourth grade. Matched control pupils who had been randomly assigned to nonacceleration achieved significantly less than those who were accelerated. In a followup study, Klausmeir (1963, 1964) found that the accelerated pupils were continuing to show no harmful effects and were achieving as well as bright children at the advanced level. Data from Rusch and Clark (1963) completely support the Klausmeier and Ripple findings at intermediate grade levels.

There is also some evidence to show that homogeneous grouping is especially effective for the upper ability group (Balow and Ruddell, 1963; Provus, 1960).

What strategies do gifted pupils use?

Namy (1967) found that gifted pupils and those misdiagnosed as gifted had similar achievement scores. The latter group apparently relied highly on memory in attaining knowledge.

What topics have proven effective with the superior pupil?

Kalin (1962) taught intellectually superior pupils a unit on equations and inequalities using both programed instruction and conventional techniques, which were equally effective.

Lewis and Plath (1959) found that high-ability children could develop generalizations about numerical relationships at a more advanced level than those normally presented to them.

As part of a long-term project, Suppes (1966) and Suppes and Ihrke (1967) reported on the use of materials on sets, coordinates systems, geometry, signed integers, logic, and symmetry.

In general, few topics have not been found to be effective with bright students.

CONTENT BY GRADE LEVEL

What content is appropriate for each grade level?

Such findings as these indicative ones are cited in studies in the following section:

(1) Study means, modes, and medians in grade 4 (Burns, 1963).

(2) Introduce geometric concepts and point set topology in grade 6 (D'Augustine, 1964) and geometric construction in grade 5 (Denmark and Kalin, 1964).

(3) Study other numeration systems in grade 1 (Scott, 1965) or grade 4 (Lerch, 1963).

(4) Study logic in grade 5 (Suppes and Binford, 1965).

The committee of Seven studied placement of topics throughout the arithmetic curriculum, and made specific suggestions for the grade placement of topics which were accepted by many schools and textbook publishers (Gillet, 1931; Raths, 1932; Washburne, 1928, 1931, 1936, 1939).

When should formal instruction in arithmetic begin?

Related to the readiness question, this has been of much concern over the years, and has been explored in some of the more recent individualized instruction studies. Postponing formal instruction until grade 5 was concluded by Sax and Ottina (1958) or grade 6 by Benezet (1936). Brownell (1960) concluded that one should begin in first grade and teach more. Neureiter and Wozencraft (1962) are among those who explored the effect of removing grade level restrictions, reporting that this resulted in higher achievement and greater interest.

GROUPING PROCEDURES

What grouping procedures have proven most effective in teaching mathematics? How effective is homogeneous grouping?

Homogeneous (ability) grouping was reported to result in favorable arithmetic achievement by Balow and Ruddell (1936), DeWar (1963), Echternacht and Gordon (1962), McLaughlin (1961), Pinney (1961), Provus (1960), Savard (1960), and West and Sievers (1960.)

Difficulty in forming homogeneous groups was noted by Balow (1964). Heterogeneous grouping was found to be more favorable for arithmetic achievement by Barthelmess and Boyer (1932) and Koontz (1961).

No differences between the two plans were reported by Davis and Tracy (1963), Holmes and Harvey (1956), Wallen and Vowles (1960), or Willcutt (DA 1957).

Individualized programs were suggested by Fawcett and others (1952), Graham (1964), Hamilton (1928), Jones (1948), Klausmeier (1964), Nabors (DA 1968), Nee (1939), Potamkin (1963), Redbird (1964), Sganga (1960), and Thompson (1941). Brewer (1966) found that teachers with "high" academic qualifications were more likely to realize the need to individualize. Availability of materials, awareness of the pupil ability range, interest, and time to plan were important factors for grouping.

How effective is Individually Prescribed Instruction (IPI)?

Generally, studies show that achievement on standardized tests is about equal to that of conventionally grouped students, while progress on IPI tests and standards is satisfactory (Bartel, DA 1966; Deep, DA 1967; Fisher, DA 1968; Scanlon, DA 1967; Yeager, 1967).

TIME ALLOTMENT

What is the most effective use of class time in elementary school mathematics?

Well-designed studies by Shipp and Deer (1960), Shuster and Pigge (1965), Pigge (1966), and Zahn (1966) reveal that maximum achievement in computation, problem solving, and mathematical concepts is obtained when over half of the time devoted to mathematical instruction is given to developmental teaching as opposed to practice. These studies reveal that pupils spending 75 percent of their time in developmental work were superior in all phases of elementary school mathematics compared with pupils spending 75 percent of their time in practice work. Hopkins (DA 1966) also supported this contention.

What is the difference in time spent on elementary school mathematics in other countries?

Miller (1958, 1960, 1962) has found that schools in foreign countries usually spend more time in the study of elementary school mathematics than do schools in the United States. Mathematical topics are introduced at an earlier level in most schools in Europe.

Is there an optimum amount of time that should be spent in elementary school mathematics? Does the amount of time vary from grade to grade?

Jarvis (1963) found that a period of 55-60 minutes produced substantially better performance than periods of 35-40 minutes. However, there is a lack of evidence in general regarding absolute amounts of time necessary to produce maximum achievement. Lawson (DA 1966) reported that fundamental skill scores were higher for a 60-minute regular group or a 40-minute concentrated group.

PLANNING AND ORGANIZING FOR TEACHING

Is the "new" mathematics superior to "old" mathematics?

The emphasis upon "new" or "modern" mathematics during the past 10 years has caused parents and teachers alike to ask this question. Clearly it is impossible to give a single definitive answer to the question since there are many types and varieties of "modern mathematics." However, the research studies cited below have delved into some phases of evaluation of current programs in elementary school mathematics.

Ruddell (1962) studied four accelerated seventh-grade classes, two of which used commonly accepted traditional programs and two which used a program of modern orientation. He found that pupils taught in the modern program scored as high or higher (statistically significant) than similar children taught in a traditional program. Simmons (DA 1966) also found that students taught under a modern program scored higher. Payne (1965) surveyed the literature and found modern programs to be as effective as traditional programs in developing traditional mathematical skills and evidence to support the conclusions that modern materials may be appropriate for a wide range of student abilities.

Hungerman (1967) compared 10 classes at the sixth-grade level who had studied the School Mathematics Study Group program during grades four, five, and six with 10 classes who had studied a conventional arithmetic program during grades four, five, and six. She found that (1) traditional achievement data (California Achievement Test) significantly favored the non -SMSG groups while contemporary achievement data (California Contemporary Mathematics Test) significantly favored the SMSG groups, (2) attitude toward mathematics was similarly positive in both groups, and (3) socioeconomic level demonstrated little or no relationship to either achievement or attitude toward mathematics. Grafft (DA 1966) found that intermediate grade pupils taught by an SMSG program understood principles of multiplication better.

Several studies occurred involving junior high school students using SMSG materials. Friebel (1967) studied six classes of pupils randomly assigned to either SMSG or the state-adopted test *Understanding Arithmetic 7* by McSwain and others. He found that the general achievement of the two groups was similar, but that the SMSG group achieved significantly superior growth in arithmetic reasoning and in concepts dealing with measurement. Cassel and Jerman (1963) studied achievement results from 262 students in grades seven, eight, and nine. This preliminary evaluation of SMSG instruction was based largely on a comparison of test scores for pupils enrolled in SMSG courses with corresponding scores for matched pupils in traditional courses. SMSG pupils had statistically significantly higher arithmetic and algebra test scores than the matched traditional pupils. Williams and Shuff (1963) studied 678 pupils in grades seven, eight, and nine and compared SMSG pupils with pupils in traditional courses. They found (1) no significant difference at the seventh-grade level, (2) significant differences at the eighth-grade level favoring the traditional groups, and (3) no significant differences in the ninth-grade groups. Osburn (DA 1966) reported no significant changes in skill development after use of SMSG materials.

Scott (1967) studied the summer loss of modern (Greater Cleveland Mathematics Program) and traditional elementary school mathematics programs. He found that, while most children suffer some summer loss in arithmetic achievement, there appears to be no systematic realtionship between the "modern" and "traditional" and students' retention of previously learned mathematical concepts.

How effective is an "activity" approach to teaching elementary school mathematics?

The current emphasis upon mathematics laboratories and the stress on correlation between science and mathematics will certainly generate research studies connected with these patterns. A somewhat similar movement occurred at an earlier point in time. The summary that follows deals with integrative activity programs. It should be noted that none of the studies described below would stand up to the present criteria for valid research.

A number of studies show results favoring an activity program. Collings (1933) found that pupils taught by an activity curriculum achieved higher scores on all arithmetic measures than pupils from a conventional subject curriculum. Harap (1934, 1936, 1937) presented findings that favored activity programs. Hopkins (1933) found that children taught in an experience curriculum achieved scores comparable to the norms established for pupils taught in a traditional curriculum. Other studies which produced results favoring activity curriculums were reported by Passehl (1949), Pistor (1934), Williams (1949), and Wrightstone (1934a, 1935b). Wilson reported evidence favoring an informal (activity) approach combined with a strong emphasis upon specific drill.

Some studies produced results unfavorable to the activity curriculum. Gates (1926) found that a systematic method resulted in higher achievement than the opportunistic method. Jersild (1939) found that groups in a non-activity program maintained a substantial advantage over those in the activity program both in arithmetic computation and arithmetic reasoning. Wrightstone (1944) found as part of an evaluation of 6 years of experimentation that the arithmetic scores of pupils in activity groups were significantly lower than those in the nonactivity groups.

What organizational patterns facilitate learning in elementary school mathematics?

Since the beginning of public education in the United States, administrators, and teachers have searched for the perfect organizational pattern. The research reported below continues this search.

Ungraded programs--A number of research studies have focused upon the use of nongraded or multigraded patterns of instruction. Finley (1963) and Metfessel (1960) found no significant difference between multigrade and single grade groups. Hart (1962) found that nongraded primary pupils achieved better in mathematics than graded groups. He dealt with only 100 pupils. In contrast Skapski (1960) found mathematics achievement to be higher in the primary graded groups. It seems safe to assume that achievement differences in mathematics are affected more by other variables than the variable of graded versus nongraded.

Team Teaching--Jackson (1964) studied 14 classes in grades five and six some team teaching and some self-contained homeroom sections. He found there were no significant differences in achievement between the two groups. The findings of Lindgren (DA 1968) were similar. Sweet (1962) surveyed pupils and teachers and found varying opinions concerning the advantages and disadvantages of team teaching in grade seven. Piage (1967) tested 300 seventh- and eighth-grade pupils, some in team teaching and some in single teacher classes. He found that team teaching appeared to be more successful at the eight-grade than the seventh-grade level. Neither grade level of pupils indicated team teaching to be the favorite form of instruction. Crandall (DA 1967) found that intermediate grade pupils achieved more in self-contained classrooms than those taught by team teaching.

Departmentalization--Periodically subject matter leaders suggest that departmentalization should be used so that the subject matter expertise of teachers can be brought into focus. Attempts to isolate the effect of departmentalization are fraught with difficulties. Thus, it is extremely difficult to conduct a valid study concerned with this topic. The findings described below should be considered in this light. Gibb and Matala (1961, 1962) studied 34 fifth- and sixth-grade classes in terms of comparing the use of special (departmentalized) teachers in science and mathematics. They found that (1) there were no significant differences in achievement between children taught in self-contained classrooms and those taught by special teachers, and (2) there was no evidence that special teachers increased pupil interest in mathematics. Gerberich and Prall (1931) found differences favoring departmentalization. It should be noted that they were dealing with a mathematics curriculum quite different from today's. Price (1967) statistically equated two fifth grades and compared departmentalization and self-contained classrooms. He found no significant differences. The findings of Grooms (DA 1968) were similar, while Eaton (1944) reported that achievement in nondepartmentalized classes was higher.

Discussion--Many studies have been conducted concerning mathematics achievement and instructional grouping. Davis and Tracy (1963) present an excellent summary of the findings of the 1950's and early 1960's, finding that studies do not reveal any clear-cut advantages for special grouping procedures.

A study of the research conducted on administrative organizational programs to meet individual pupil needs is inconclusive. A proponent for one plan can find studies that verify his stand. Conversely, an opponent of the same program can find studies that show that his plan works no better than the typical administrative, single teacher, graded pattern. Perhaps the most important implication of the various studies is that good teachers are effective regardless of the nature of classroom organization.

METHODS OF INSTRUCTION

How effective is the "meaning" method?

Since the early 1930's, mathematics educators have advocated that "pupils should understand the mathematics they are taught." This goal gave rise to the "meaning approach" to teaching elementary school mathematics. Typically the meaning approach is contrasted with the rote learning or rule approach in which the pupil does not develop an understanding of the rationale of the mathematics he is taught. Certainly the meaning approach laid the foundation for "modern mathematics."

The majority of studies which involve the meaning approach are remarkably consistent in their findings. Typically researchers found that (1) rote rule and meaning produce about the same results when immediate computation ability is used as a criterion, (2) when retention is used as a criterion the meaning method is superior to the rote rule method, (3) greater transfer is facilitated by the meaning method, and (4) the meaning method produces greater understanding of mathematical principles and comprehension of complex analysis. (See: Brownell 1949; Dawson, 1955; Greathouse, DA 1966; Krich, 1964; Miller, 1957; and Rappaport, 1958, 1963). Specific findings for use of this method can be found under the sections dealing with the mathematical topics taught in the elementary school.

How effective are "discovery type" of teaching approaches compared to "expository type" teaching approaches?

A number of good studies have been addressed to this question. An excellent study by Worthen (1968) with 432 pupils at the fifth- and sixth-grade level compared discovery and expository presentation. From his findings he suggests that (1) if pupil ability to retain mathematical concepts and to transfer the heuristics of problem solving are valued outcomes of education, discovery sequencing should be an integral part of the methodology used in presenting mathematics in the elementary classroom, and (2) if immediate recall is a valued outcome of education, expository sequencing should be continued as the typical instruction practice used in elementary classrooms. It is suggested that the Worthen study is well worth reading by all interested in discovery-type teaching.

Henderson and Rollins (1967) found three types of inductive (discovery) strategies to be effective in teaching concepts and generalizations. Armstrong (DA 1968) reported that the inductive mode fostered the learning of operations, while the deductive mode resulted in greater learning of mathematical properties. Meconi (1967) used programed materials to compare rule and example, guided discovery, and discovery techniques at the eighth- and ninth-grade level. He found that pupils learned effectively with each technique. The findings of Hanson (DA 1967) were similar.

Scandura (1964a, 1964b, 1964c) conducted several related studies concerned with exposition versus discovery. He found that (1) discovery pupils were better able to handle problem tasks, (2) the discovery group took longer to reach the desired level of facility, and (3) exposition pupils generally used the algorithm taught, while discovery subjects seemed more reliant.

STUDENTS AND TEACHERS

EFFECT OF TEACHER BACKGROUND ON STUDENT ACHIEVEMENT

What effect does the background of the teacher have on student achievement?

It is true that one cannot teach what one does not know. It also seems true that teachers of elementary school mathematics who have studied mathematics for some time or in great depth should be able to bring their experience to the classroom resulting in greater achievement by those students thus exposed. Bassham (1962) found that this was true. Teachers with more experience in mathematics had pupils with greater achievement in mathematics. Shim (1965) supported this finding insofar as the measurement of teacher experience in mathematics was not in terms of grade point average, time in college, length of certification, etc. In general, teachers with a greater understanding of mathematics were able to share that understanding with their pupils.

What effect does inservice education for teachers have on student achievement?

Studies by Houston and DeVault (1963), Ruddell and Balow (1963), Ruddell and Brown (1964, and Hurst (DA 1968) all confirm the fact that teachers involved in inservice education in elementary school mathematics are able to bring this experience to the classroom, resulting in greater achievement by their pupils. Studies by Rouse (DA 1968) and Lampela (DA 1966) were in disagreement. Scaramuzzi (1956) found that teachers who are able to apply their imagination to the solution of problems in motivating elementary school mathematics also found a greater level of pupil achievement.

INSERVICE TRAINING

What is the most effective way to conduct inservice courses?

Brown (1965) evaluated one approach using consultants and workshops, and found it increased understanding and use of new techniques. Creswell (1967) found that workshops did not appear to be sufficiently effective, but Whitman (1966) reported increased scores in conceptual knowledge. Dutton and Hammond (1966) found a course using a college professor and a textbook less effective than one using district staff and a variety of instructional materials. They suggested that the second program was less structured, but more adapted to individual needs. Classroom consultant services apparently were useful (DeVault, Houston, and Boyd, 1963). Harper (1964) reported increased achievement, and Todd (1966) reported increase in both achievement and favorable attitude after a "mathmatics for teachers" course. Weaver (1966) found teachers who had been exposed to geometry scored higher on a geometry inventory. Beers (DA 1968) found discussion alone to be more effective than when combined with supervised study, and Foley (DA 1966) found teachers achieved as much in a large class as in smaller classes with discussion. Correspondence courses using television and programed materials were effective, according to Green (DA 1968). Lindsay (DA 1966) found both lecture-discussion and programed courses were effective. Kennedy and Alves (1964) surveyed teachers, and found wide variability in their suggestions. The most agreement was expressed for courses which combined content and methods.

PHYSICAL, PSYCHOLOGICAL, AND/OR SOCIAL CHARACTERISTICS OF STUDENTS

How do personality factors affect achievement?

Under-achievement has been related to personal adjustment and is often considered as influential in relation to achievement in arithmetic as intelligence is. Various aspects and degrees of adjustment have been investigated in relation to arithmetic achievement with some interesting results. A study by Capps (1962) found retardation in arithmetic tended to be related to personal adjustment. and positive correlations between arithmetic achievement and a healthy personality were found by Cleveland and Bosworth (1967). Wilson (1959) had contradictory results in that no certain differences in arithmetic achievement were found for pupils who scored at or below the tenth percentile on a personality test, when compared to pupils who scored at the 50th percentile. Ridding (1967) found extraversion correlated with over-achievement and introversion correlated with under-achievement. A related study by Buswell (1953) found that, when intelligence was controlled, status of social acceptability was not related to achievement.

Children classified as emotionally disturbed were found to have lower arithmetic scores than reading scores in two studies, one by Stone and Rawley (1964) and one by Tamkin (1960).

The relationship of delinquency or social maladjustment to arithmetic achievement has been investigated in several studies. Socially maladjusted boys showed poorest achievement in the area of arithmetic (Feinberg, 1947) and delinquent below-average IQ children performed better on nonverbal intelligence tests than on verbal intelligence tests (Richardson and Saerko, 1956). Dinitz and others (1957) found delinquent-prone boys had significantly less arithmetic competence than nondelinquent-prone boys, and an older study by Lane (1934) found that delinquent boys' poorest achievement was in subject areas which required drill, as in arithmetic computation.

Are there any sociological characteristics that distinguish pupils of varying mathematical ability?

In the past, the one-room schoolhouse was a common educational situation. In the 1930's, the relationship of achievement to the sociological characteristics of rural or community schools was investigated by McIntosh and Schrammel (1930) and Clem and Chester (1933). Both studies investigated achievement in rural schools as compared to village or graded schools, and both found that those in village or graded schools had higher arithmetic achievement.

Some older studies, concerned with cultural characteristics which are still prevalent in today's society, compared achievement of white children to achievement of Mexican children (Coers, 1935) and American Indian children (Hansen, 1937). The white children had higher arithmetic achievement scores in both studies; but when Coers considered mental ability, Mexican children were found to be achieving more for their measured level. A study by Manuel (1935) found Spanish-speaking children had higher arithmetic achievement scores than English-speaking children, but the reverse was true of reading achievement scores. Harris (DA 1968) reported that Negro children achieved less well than white children, but did better in arithmetic than in most other areas.

Does handedness have an affect on arithmetic achievement?

Various physical characteristics of pupils have been investigated to see if they differ with different levels of mathematics achievement. One physical characteristics that has been investigated, mainly in relation to reading, is handedness. Groff (1962) carried on an investigation of the relationship of hand preference to arithmetic achievement, and found some differences that indicated that left-handed pupils had lower reasoning scores. He points out that other factors may have accounted for the differences found in his study.

Early elementary teachers are constantly aware of pupils reversing letters and numbers when writing. A study that deals with confusion or nondominant handedness as a possible explanation of reversals was done by Zaslow (1966). He found that having children move the hand and arm so it crossed the body in mid-line resulted in significant corrections of reversed numbers and letters.

To what extent do siblings resemble each other in intelligence and mathematical achievement?

A study by Schoonover (1956) found a substantial relationship between the intelligence and achievement for siblings. He found that sisters were more similar to each other in arithmetic achievement than in other achievement areas.

What are the achievement characteristics of children from orphan or foster homes?

Two studies by Feinberg (1949, 1954) were concerned with achievement of children from foster and orphan homes as compared to achievement of maladjusted children. Children from foster homes achieved more than children from orphanages. Both groups achieved more than maladjusted children, but arithmetic was found to be a difficult subject for them as a group.

PRESERVICE TRAINING

What are effective procedures for preservice preparation?

Mathematics courses and methods courses resulted in increased understanding of concepts and attitudes reflecting a growing appreciation of arithmetic (Dutton, 1961, 1965, 1966; Dutton and Cheney, 1964; Smith, 1967; Weaver, 1956). Strong indication of which type of course is best is lacking, though separate methods and content courses (Wickes, DA 1968), a combined content- methods course (Phillips, 1968), a CAI course (Riedesel and Suydam, 1967), and a remedial course (Dutton, 1966; Waggoner, 1958) were shown to be effective. Gibbons (DA 1968) reported that discussion classes were more effective than those without discussion, while Northey (DA 1967) could not find that any proportion of time for lecture or discussion was better than any other. Use of enrichment problems was helpful (Litwiller, DA 1968). Bassler (DA 1966) used exercises which were either purely mathematical or framed in a physical world setting, but found no resulting difference in achievement.

How mathematically competent are preservice teachers?

The majority of the studies in this category were surveys, and reflected surprisingly similar conclusions over a period of years. In summary, they showed that the mathematical competency of preservice teachers:

(1) is inadequate (Creswell, 1964; Fulkerson, 1960; Glennon, 1949; Reys, 1968; Skypek, 1965; Smith, 1963; Taylor, 1938; Weaver, 1956; Callahan, DA 1967)

(2) seems to be improving (Clark, 1955; Combs, 1963; C.U.P.M., 1967; Fisher, 1967; Grossnickle, 1962; Rappaport, 1958; Jensen, DA 1967; Withnell, DA 1968)

What are the attitudes of preservice teachers?

Attitudes of preservice teachers toward mathematics were:

(1) majority, unfavorable (Dutton, 1951; Smith, 1964)
(2) slightly more favorable in 1962 than in 1954 (Dutton. 1962); slightly more favorable after mathematics preparatory courses (Reys and Delan, 1968; Gee, DA 1966)
(3) favorable (Kane, 1968)

Unfavorable attitudes were related to lack of understanding, disassociation from life, boring aspects, insecurity and fear of making mistakes, and difficulty (Dutton, 1951, 1954, 1962).

Favorable attitudes were related to enjoyment, importance, challenge, and good teachers (Dutton, 1951, 1954).

SEX DIFFERENCES

What differences in mathematical achievement can be attributed to sex?

It should be noted that differences related to sex are not limited to mathematical achievement in elementary school. There is also a distinct difference between pre-junior high achievement and achievement of those of junior high school and beyond. Almost all of the related research indicates that pre-junior high school girls achieve more than pre-junior high school boys except in arithmethic. Studies by Heilman (1933), Stroud and Lindquist (1942), Powell (1963), and Jarvis (1964) all support this indication and show no significant differences between the sexes in arithmetic achievement. From junior high school and beyond the research indicates the same superiority of girls in general, but boys now surpass girls in studies involving science and mathematics. Studies by Blackwell (1940), Alexander (1962), Wozencraft (1963), and Powell (1964) support this view.

What differences in mathematical achievement are related to self-concept?

In many cases it is not as much ability that determines achievement as the student's concept of his ability: "How well should I be doing in relation to the other students?" There seems to be some indication in the studies by Unkel

(1966) and more especially Clark (1967) that girls do not show superior achievement in arithmetic, science, and mathematics simply because they feel that girls should not show superior achievement in those fields. Boys, on the other hand, may feel that subjects other than science and mathematics are not masculine enough, not rough and ready enough, for them to show superiority. Certainly adjustments need to be made in the mathematics curriculum to accommodate girls as well as boys.

SOCIOECONOMIC DIFFERENCES

Can differences in mathematical achievement be attributed to differences in socioeconomic environment?

One of the most important topics for discussion in education today is the topic regarding socioeconomic environment and achievement. Studies by Montague (1964), Dunkley (1965), Dutton (1967), Binkley (DA 1967), Searle (DA 1968), Skypek (DA 1967), and Unkel (1966) reveal that there is a high correlation between socioeconomic environment and achievement and that the lower the level of economic environment the lower the elementary school mathematics achievement. This relationship seems to indicate that children from low socioeconomic backgrounds have a scholastic handicap in direct proportion.

Can differences in mathematical achievement due to socioeconomic differences be reduced?

It seems logical that to reduce the differences in mathematical achievement due to differences in socioeconomic environment would be to reduce the differences in the socioeconomic environment. Since this seems to be impossible, attempts have been made to reduce the effects of the environment. Paschal (1966) and Newman (1967) found that by recognizing the handicap that a low socioeconomic environment places on a pupil they could, by paying special attention and giving great amounts of individual assistance, reduce the differences in achievement by increasing the achievement of these individual pupils. Pitts (1968) found more success in reducing the environmental handicap by providing preschool experience to as many of the children from low socioeconomic background as possible. This program was similar to the Project Head Start. Hollander (DA 1968) reported gains in speed and accuracy when verbal praise and candy rewards were given to sixth-grade inner city children.

STUDENT ATTITUDE AND CLASSROOM CLIMATE

Do elementary pupils like mathematics?

It is a widely accepted notion that mathematics is disliked by most pupils; however, results of numerous surveys contradict this notion. Many studies provide results which show that pupils frequently select arithmetic as their favorite subject (Inskeep, 1965; Mosher, 1952; Rowland, 1963). Several other surveys report arithmetic as being above average as a preferred subject (Anderson, 1958; Chase, 1949; Curry, 1963; Herman, 1963; Stright, 1960; Greenblatt, 1962). Chase (1958), Curry (1963), and Dutton (1956) found middle-grade boys rating their liking for arithmetic slightly higher than girls; but Stright (1960), when including lower grades, found girls showing a slightly higher preference. Chase (1949) reported that New England pupils rated arithmetic slightly higher than pupils in the Southwest. Dutton (1956) found pupils to report lack of understanding,

difficulty, poor achievement, and boring aspects of arithmetic as major reasons for their dislike of arithmetic.

Do pupils show a preference for modern or traditional mathematics?

Generally, it has been found that pupils who like mathematics like either modern or traditional programs. Abrego (1966) compared pupil attitudes toward modern versus traditional mathematics and found that pupils who like one type, liked the other. Hungerman (1967) found that pupils hold positive attitudes towards both conventional and contemporary mathematic programs. Dutton (1968) reported a slight increase in attitudes towards modern mathematics when compared with pupil attitudes of 10 years before.

How does the attitude of the teacher affect the attitude of the pupil?

This question cannot be answered directly, but the relationship of the teacher and pupil attitudes has been investigated, with differing results. Inskeep (1965) found no relationship between teacher and pupil attitudes, but Chase (1958) found that the pupils of teachers who preferred arithmetic appear to favor it themselves. With high intelligence pupils, Greenblatt (1962) reported that the preference of teachers corresponded strongly with that of the pupils.

How does the classroom climate affect pupil learning in mathematics?

The influence of differing classroom climates on arithmetic achievement has been investigated by Guggenheim (1961), who found no significant differences for classrooms that were and were not dominated by the teacher. The amount and kind of interaction was investigated by Hudgins and Loftis (1966), who found that teachers initiated interaction more frequently with average-ability pupils than with high-ability pupils.

What procedures improve pupil attitudes towards mathematics?

When arithmetic is taught as a skill that has practical value and is useful in out-of-class situations, attitudes become more positive. Studies by Dutton (1956), Lyda and Morse (1963), Malone and Freel (1954), and Stokes (1956) reached conclusions to support this statement. Stokes (1958) found higher sustained attention of pupils and Hunnicutt (1944) found activity methods associated with awareness of out-of-class use of arithmetic. Fedon (1958) found a positive increase in attitude when problem solving was related to experiments. Both presentation of arithmetic by television (Kaprelian, 1961) and specific review (Burns, 1956) seemed to create more positive attitudes.

What is the relationship between achievement, ability, and attitude?

The contribution of attitude and interest to achievement is not easily measured because of other variables; but research by Bassham, Murphy, and Murphy (1964), Dean (1950), Lyda (1963), Powell (1966), and others indicates there is a positive relationship. Anttonen (DA 1968) is among those who found no relationship. Greenblatt (1962) found girls with high arithmetic achievement had more positive attitudes. Rowland and Inskeep (1963) found a feeling of success

increased preference and attitudes for arithmetic. The relationship of intelligence, which cannot be disassociated from achievement, was investigated by Rice (1963), Greenblatt (1962), and Stephens (1960), who found gifted or accelerated pupils had a higher interest in arithmetic. A study having related findings is one by Feldhusen and Klausmeier (1962), in which a significant relationship was found between high anxiety and low arithmetic achievement for low IQ pupils.

STUDENT USE OF ARITHMETIC

What mathematics is used by pupils outside the classroom?

Ellsworth (1941) found that in an urban area children used telling time, money, counting, and reading numbers most frequently, while measuring area and operations with fractions were used least often. Moseley (1938) found the order of use at the sixth-grade level was money, subtraction, addition, multiplication, measuring, division, and fractions, with games, shopping, and chores providing the greatest occasions for use. Smith (1924) found that first graders used arithmetic in stores, in games requiring counting, and in reading Roman numerals on the clock and Arabic numerals on book pages. Addition and counting were most frequently used at this grade level. Addition was also used most by third graders (Wahlstrom, 1936), and division very rarely used. Willey (1943) ordered the uses as money, measurement, time, objects, pets and distance, finding counting, fractions, and subtraction were most often needed in problems.

TEACHER ATTITUDES

How do teachers feel about teaching mathematics?

Brown (1965) noted that, while teachers feel inadequate in teaching mathematics, they still like to teach it. Bean (1959) found that teachers did not perceive themseleves as competent after taking a mathematical understanding test as they had before it. Barnes, Cruickshank, and Foster (1960) reported that teachers who were judged superior tended to underrate themselves, while those judged fair tended to overrate themselves and had a more negative attitude toward mathematics. Turner and others (1963) reported several studies with the Mathematics Teaching Tasks Test, on which high scores were found to be related to high pupil achievement.

Hollingsworth, Lacey, and Shannon (1930) reported that teachers at that time thought arithmetic and reading were the easiest subjects to teach, because of (1) personal liking, (2) thorough knowledge and training and (3) adequate tests and organized courses. Huettig and Newell (1966) reported that teachers with more than 10 years of experience were less positive toward a modern mathematics program, while positive statements increased with the amount of training.

TEACHER COMPETENCY

How competent are teachers to teach mathematics?

Teachers were found to be weakest in whole number, decimal, and percentage concepts (Kenney, 1965). Few processes, concepts, and relationships were understood by the majority of teachers (Orleans and Wandt, 1953; Robinson, 1935). LeBaron (1949) reported that only half of the teachers responding expressed agreement with research findings.

Stoneking and Welch (1961) reported that amount of preparation was reflected in higher scores more than age or teaching experience were, but Hand (DA 1c 7) found experience was a sig ficant factor. Buck (DA 1968) failed to observe differences in teaching benavi .s due to mathematics achievement or classroom experience, nor did Dickens (DA 1966) observe changes after a course, despite increased achievement.

Griffin (DA 1967) surveyed over 1,000 teachers and found that they understood only half of the total topics and one-third of the modern topics. Williams (DA 1966) also reported low levels of achievement when compared with pupils; and Kipps (1968) cited details, resulting from an inventory, of what teachers understand about mathematics.

TEACHING METHODS AND STRATEGIES

CHECKING

How effective is checking as a procedure to reduce errors in mathematical problems?

Every mathematics teacher, at one time or another, has said, "Be sure to check your work when you have finished." Every student, at one time or another, has indeed checked his arithmetic problems for errors; and every student has been surprised to find that there were still errors in his work after checking. Grossnickle (1935, 1938) reported that checking is an ineffective procedure to reduce computational errors in division and subtraction. He found that, if the student's check revealed some discrepancy, the student would force the check to that of the answer he produced for the problem. Karstens (1946) found that only in problems of estimation where a certain particular check was useful was any accuracy attained in checking procedures. It may be true that computational errors are mostly errors in understanding either the computational procedure or the underlying assumptions, or both. In that case, the check is another computational procedure to be misunderstood. Also, the lack of accuracy in checking may be related to the pupil's not sensing a reason to check.

Does checking answers result in improved achievement?

Clark and Vincent (1926) found that checking answers resulted in greater accuracy, especially when the number of problems attempted was considered. Thus, the technique of giving pupils fewer exercises, but having them check their answers, is suggested.

DIAGNOSIS

What are the most common errors made by pupils? (What are the most common misconceptions that pupils have concerning mathematical understanding?)

It was generally agreed that errors with combinations were the most frequent source of error. In an extensive diagnostic study (Buswell, 1926), various poor work habits were cited for each operation. Many of these, however, were related to the teaching procedure and are no longer completely appropriate. Nevertheless, errors with combinations were most frequently cited. Specific remediation based on diagnosis of the errors was found to be fairly successful.

Smith and Eaton (1939) found addition facts were most thoroughly mastered at the fourth-grade level, with zero combinations most frequently missed.

In analyzing errors with fractions, Brueckner (1928a) found 21,065 errors, of which the major ones were computational. Lack of comprehension of which process was involved, inability to express fractions in their simplest form, and difficulty in renaming improper fractions were also causes of error. Morton (1924) and Shane (1938) substantiated these results. Scott (1962) found regrouping errors with subtracting fractions were more frequent than in whole number work.

More errors of this type were found with children using a contemporary program than Brueckner noted in 1928.

Brueckner (1928b) found 114 different kinds of errors with decimals; most common was misplacement of the decimal point. Guilder (1946a) reported that changing fractions to decimals, renaming mixed numbers, and division with decimals were the greatest sources of difficulty.

For addition and multiplication, Burge (1932, 1934) reported that errors with combinations and carrying were most frequent. Knight and Ford (1931) noted that the later a multiplication fact appeared in an example, the more frequent were errors with it; but Wilson (1936) disputed this.

Grossnickle (1934, 1935, 1936a, 1936b, 1939, 1941, 1943) analyzed division errors, reporting that combination errors were most frequent (38.8%), while difficulties with remainders accounted for almost one-fourth of all errors. Errors with zero facts were constant across all operations (Grossnickle and Snyder, 1939).

In work with percentage at grade nine, Guilder (1946b) reported that at least half of the pupils had difficulty, with almost everyone unsuccessful at finding a number when a percent of it is known.

Lutes (1926) found that errors on verbal problems resulted more from computation than from ignorance of a principle or rule or from lack of comprehension. Morton (1925) reported that use of incorrect procedures accounted for over half of the errors. Errors with addition and subtraction in word problems were less frequent than those with other operations (Ross, 1964). Roberts (1968) analyzed third-grade test papers, and categorized four types of errors: wrong operation, computational, defective algorithm, and undiscernable, with defective algorithms accounting for the largest number of errors.

How can errors be most effectively diagnosed?

Brownell and Watson (1936) and Burge (1934) reported that use of an interview technique was more reliable in ascertaining errors than a test was. Brueckner (1928a, 1928b) and Brueckner and Elwell (1932) counted errors with fractions and decimals made in written work. Grossnickle (1935) reported that he found that at least three responses to each fact must be made by pupils for diagnosis to be reliable. It was suggested by Olander (1933) that teachers diagnosed more accurately in division than in the other three processes. Aftreth (1957, 1958) reported that systematic analysis of errors in the study of fractions was not particularly helpful, while Dougherty (1962) presented a more successful program in which pupils diagnosed their own errors. Guilder (1936) used individualized group remedial work; Harvey (1935) suggested specific provisions for reteaching. Eaton (1938) used a dictaphone to record verbal responses successfully.

HOMEWORK

Does homework increase pupil achievement in elementary school mathematics?

Though assignment of homework is an accepted practice in many mathematics teaching situations, the value of homework is frequently questioned by teachers parents, and pupils. Studies concerning the effect of homework on mathematics achievement are limited, and research on the effect of homework on achievement is confounded by a host of variables. Generally, the studies before 1960 do not show consistent results in terms of improved pupil achievement (Foran and Weber 1939; Goldstein, 1960; Steiner, 1934; Teahan, 1935; Vincent, 1937). In a recent study, Koch (1965) found no difference in problem solving achievement, but significant improvement in concept achievement. Maertens (DA 1968) reported no significant differences between types of homework, as did Whelan (DA 1966).

What type of homework seems most effective?

Few studies investigate variables related to mathematics homework. Koch (1965) found that with sixth graders both full or half homework assignments resulted in significant achievement of arithmetic concepts. Steiner (1934) found arithmetic homework in terms of achievement. Slow sixth-grade pupils showed greater gain than average pupils in a study by Vincent (1937). An individualized method was favored by Bradley (DA 1968).

PROGRAMED INSTRUCTION

How effective is programed instruction in teaching mathematics?

Results from studies using various ways of presenting programed material show differing results. Banghart, McLaulin, Wesson and Pikaart (1963), Brinkmann (1966), and Fincher and Fillmer (1965) found that pupils having instruction via various methods of programed instruction, as compared to conventional instruction, made significant achievement gains; but Arvin (DA 1966), Donaldson (1968), Feldhusen, Ramharter, and Birt (1962), Meadowcroft (1965), and Spagnoli (1965) found no significant differences. Pupil attitudes were found to be more favorable toward programed instruction by Feldhusen and others (1962), but Meadowcroft (1965) found accelerated pupils having more favorable attitudes toward a method using the least amount of programed instruction, and Brinkmann (1966) found that pupils who were below the median in achievement favored teacher instruction.

How effective are various methods of presenting programed instruction?

Programed instruction can be presented in a variety of ways. Eigen (1962) found no significant difference when materials were presented by teaching machines, vertical text, or horizontal text; and Higgens and Rusch (1965) found no differences for programed textbooks versus a workbook for remedial teaching. Miller (1964) found written plans plus automated practice machines superior to textbooks with concrete materials in achievement gains. A study by Crist (1966) found no difference in individual or group-paced use of programed texts.

Austin (DA 1966) found that both constructed responses and multiple choice responses were effective.

How can programed instruction be most effectively used as part of the teaching process?

Of the infinite number of ways that programed instruction could be used with teacher instruction, few combinations have been investigated and reported in research literature. Programed instruction during teacher instruction, as contrasted with preceding and following teacher instruction, was investigated by Meadowcroft (1965). This study found more positive attitudes for all groups, but higher achievement for the average group who had programed material during teacher instruction.

What type of pupils seem to benefit the most from programed instruction?

The use of programed instruction with mentally retarded children (Blackman and Capobianco, 1965) resulted in significant behavior change but not significant achievement when compared to conventional methods. Kalin (1962) found that programed materials did not produce superior achievement with high IQ pupils. However, less time was needed to finish materials. In contrast, Fincher and Fillmer (1965) found high IQ pupils performed better with programed instruction. Traweek (1964) found no significant difference for IQ, but concluded that programed instruction may be a promising method of teaching poorly adjusted students.

What mathematical content has been taught with programed materials in research situations?

Frequently an experimenter will select a topic that pupils would normally have little knowledge of, thus adding control in terms of the limited scope of initial knowledge. Geometry topics--including topology, sets, relations and functions--have been used by Brinkmann (1966), D'Augustine (1966), Denmark and Kalin (1964), Gagne and Bassler (1963), and Randolph (1964). Advanced topics were used by Kalin (1962) and latitude and longitude by Spagnoli (1965). Various operations with fractions were used by Greatsinger (DA 1967), Krich (1964), Levin (DA 1968), Miller (1964), and Traweek (1964); and Eigen (1962) used numbers and numerals. General lower-grade arithmetic was used by Banghart and others (1963) and by Fincher and Fillmer (1965). Remedial multiplication and division were studied by Higgins and Rusch (1965). Riggs (DA 1967) developed a text to interpret graphs.

How effective is CAI in teaching mathematics? How can CAI be effectively used?

Suppes has reported (in various progress reports for the Stanford Project) success in using both drill and practice and tutorial computer-assisted instruction programs at the primary grade level.

TESTING

What procedures are most effective in testing computational skills? understanding?

Brueckner and Hawkinson (1934) found that grouping types of items on one test resulted in better achievement on a second test where types were not grouped. Capron (1933) found no difference in number of process errors on tests in which problems were arranged in random order, from easy-to-hard, or from hard-to-easy. In testing of division of decimals, Grossnickle (1944) found random sequence of items was more difficult than when items were grouped by type. The "atmosphere" of the test situation was found to be a significant factor by Goodwin (1966). The interview technique propounded by Brownell (1936) was modified by Gray (1966). Hartlein (1966) found coded items to be effective, while Graham (DA 1967) used scalogram analysis.

What types of tests are reported?

In research reports, development of the following types of tests have been reported:

(1) Readiness for division (Brueckner, 1940)
(2) Readiness for first-grade arithmetic (Brueckner, 1947; Hildreth, 1935; Ferguson, DA 1967)
(3) Readiness for signed numbers (Olander, 1957)
(4) Readiness for fractions (Souder, 1943)
(5) Vocabulary (Chase, 1961)
(6) Problem solving (Connor and Hawkins, 1936)
(7) Fundamentals (Courtis, 1909, 1911; Foran and Lenaway, 1938; Olander, Van Wagenen and Bishop, 1949)
(8) Understanding (Van Brock, 1965; Ashlock, 1968: Ashlock and Welch, 1966; Flournoy, 1967a, 1967b, 1968; Hartlein, 1966)
(9) National survey tests (Romberg and Wilson, 1968)
(10) Geometry (Weaver, 1966)
(11) Arithmetic principles (Welch and Edwards, 1965)

In addition, of course, tests were developed as one aspect of many other studies.

MISCELLANEOUS

FOREIGN COMPARISONS

How do children in the United States compare with children in foreign countries in elementary school mathematics?

There has always been a great deal of discussion regarding the performance of educational systems in foreign countries producing talented intellectuals, with the obvious implication that the educational systems throughout the United States do not. This was especially evident in the mid-1950's with the advent of Sputnik I. However, research studies directly comparing children from the United States and some foreign country are either rare or poor. Buswell (1958) reported that students in England in grades 5 and 6 were superior to students in the same grades in the United States. However, studies by Bogut (1959) and Pace (1966), using virtually the same data, concluded that this difference was attributed to the additional year of education that English children have up to that level. Johnson (1964) found that United States children were superior to English children as measured by an achievement test from the United States, while the opposite was true when the groups were tested by an English test. The attitude of American students was found to be more positive than that of London students by Johnson (DA 1966). Cramer (1936) revealed that United States children were superior to Australian children on an Australian test in grades 4 and 5, but just the opposite in grades 6 and 8. Wilson (1958) found no differences between United States and Canadian children in grades 2 and 3. Kramer (1959) found Dutch children superior to a group of children from Iowa. In general, children in foreign countries do as well as children in the United States as measured against their own culture, by their own measures.

Do foreign countries place a greater emphasis upon mathematics than does the United States?

In order to evaluate emphasis, many researchers have examined and compared foreign textbooks, curriculum, and topic length. Sherman (1965) reported that the mathematics curriculum in Russia placed many mathematics topics at lower levels than the United States did and introduced more topics into the elementary school mathematics curriculum than the United States did. He concluded, however, that all of the topics were disconnected and discontinuous. Brownell (1960) reported on education in Scotland and concluded that children are able to handle mathematical topics earlier than now seems feasible, and that the attention span of children is longer than now thought. Miller (1960, 1962) reported for numerous European countries that elementary school mathematics topics, such as geometry, are introduced early in the curriculum, and that the textbooks contain more time for more rigorous practice. Dominy (1963), McKibben (1961), and Shutter (1960) all report that there seems to be little difference in achievement between countries. They also confirm the report that mathematical topics are introduced earlier in the curriculum than in the United States. Mehl (DA 1966) found that French textbooks placed more stress on problem solving.

What are some of the studies of educational systems of foreign countries?

There are many studies which discuss the elementary school mathmatics programs of foreign countries without trying to compare them to those of the United States. They do not attempt to relate some of the advantages and disadvantages to the curriculum of the United States. They are merely reporting the ideas expressed or the materials used in the specific country. DeFrancis (1959) and Vogeli (1960) reported about Russia. Pella (1965) and El-Naggar (DA 1966) discussed the Middle East, while Zur (DA 1968) cited implications for Israeli. Buell (1963) examined Sweden. Sato (1968) studied Japan. Wirszup (1959) reported on Poland and other communist countries, as Fehr (1959) did with 16 other European nations. Dutton (1968) studied the elementary school mathematics system of Ethiopia and recommended considerable changes be made in teacher education as well as in texts. Bruni (DA 1968) reported on recent Italian experimentation. There is a lot that can be learned in studying the educational systems of others in terms of ideas and materials used, but comparison can be made only with extreme care.

NUMERATION SYSTEMS (Ours and Others)

What materials are most effective in teaching place value?

Lyda and Taylor (1964) found that instruction on modular arithmetic did not result in greater understanding of our numeration system than the regular program did. Pupils who were taught place value concepts through the use of a ruler achieved a median retention score of 70 percent (Johnson, 1952).

What are the most common errors made by pupils?

In a survey, Flournoy, Brandt, and McGregor (1963) found that errors related to (1) the additive principle; (2) relative interpretations; (3) the meaning of 1,000 as 100 tens, 10 hundreds, etc.; (4) expressing powers of 10, as 10,000= 10 x 10 x 10 x 10; and (5) the 10-to-1 relationship in place value.

Is there transfer from historical systems to better understanding of our system?

Bradley and Earp (1966) found that few teachers stress underlying principles. Schlinsog (DA 1966) reported no significant effects of instruction on other number bases, while Scrivens (DA 1968) found that teaching about Egyptian numeration was more effective than teaching about base five numeration. Smith (DA 1968) reported that study of nondecimal systems produced effective achievement and retention, but little effect on decimal system understanding was found.

What are the most effective methods of teaching other bases?

Use of a variable base abacus was not found to result in greater achievement than use of the chalkboard (Jamison, 1964). A story about the use of a number base among a mythical group of people was effective, according to Lerch (1963).

How much transfer to base ten does the teaching of other bases have?

Lerch (1963) reported that increased understanding of base ten resulted from teaching base five. However, in a carefully conducted study, Schlinsog (1968) examined the effects of nondecimal instruction on basic understanding, computational ability, underachievement, and preference, and found no significant differences from regular decimal-base instruction.

Hebron (1962) did a factorial study of items and found that knowledge of one system is the most important single factor in learning a new one. Jackson (DA 1966) reported that pupils receiving instruction in nondecimal numeration systems did significantly better in tests measuring understanding and problem-solving skills but not on computation than those studying the decimal system.

At what grade level can other bases be most effectively introduced?

Scott (1963) reported that first graders outperformed kindergarteners. Lerch (1963) and Hollis (1964) reported successful use of other bases in grade four.

APPLICABLE GENERALIZED FINDINGS FROM--- ELEMENTARY MATHEMATICS RESEARCH

No. 11-B

These are findings which the authors believe to be clearly substantiated by the research on elementary school mathematics. The items are not drawn from any one study but are generalizations from many. They seem applicable to the modern mathematics curriculums, often across a wide range of grades; they may also be applicable to other subject areas.

Concept Development and Learning

. Instruction in arithmetic should be based on the readiness of pupils.

. Meaningful teaching increases retention, transfer, and understanding.

. Modern mathematics programs tend to produce better reasoning and retention, but do not improve computational skills.

. Teaching for transfer is necessary.

. Transfer is greatest when content is similar.

. New concepts introduced at the end of the school year are less likely to be retained over the summer vacation.

. Rate of learning is related to intelligence.

. Intelligence is related to achievement.

. Periodic review increases retention.

. Immediate review of arithmetic test items increases achievement and retention.

. Reinforcement increases achievement in mathematics.

- Children know a great deal of mathematics before they enter kindergarten.

- Developmental stages (such as Piaget's) appear to be related to mathematical achievement.

- Many children can count by ones to 10 and beyond upon entering kindergarten.

- Arithmetic achievement is related to reading ability.

- Motivation is important for arithmetic achievement.

- Verbal praise aids motivation and achievement.

Materials

- Use of mathematical games increases motivation.

- Concrete materials should be used before proceeding to abstractions.

- The reading level of many arithmetic textbooks is too difficult.

Mathematical Areas

- Counting money and telling time are the most frequent out-of-school uses of arithmetic skills by pupils.

- For legible numeral writing, continuous emphasis is necessary.

- A variety of problem-solving procedures should be systematically taught.

- Specific training in mathematical vocabulary increases problem-solving ability.

- Problems of interest to pupils promote greater achievement in problem solving.

- Characteristics of good problem solvers include higher intelligence, strong computational skills, ability to estimate and analyze, skill in noting irrelevant detail, and understanding of concepts.

Mathematical Operations

- Drill and practice are necessary for computational accuracy.

- Drill should be used only after effective developmental activities.

- Drill should be spaced and varied in type and amount.

- Practice in mental computation should be provided.

- Proficiency in counting facilitates in learning of addition.

- Computational errors with basic facts are the greatest source of pupil difficulty, with lack of understanding second.

- In all four operations, basic facts vary in difficulty.

Organization for Instruction

- A systematically planned program of instruction in arithmetic is better than incidental instruction.

- The type of classroom organization (departmentalized, team teaching, self-contained, etc.) apparently does not affect achievement.

- Grouping is desirable, especially within a class.

- Individualizing instruction improves immediate achievement, retention, and transfer.

- At least one-half of the class time should be spent on developmental activities.

Students and Teachers

- Elementary school pupils generally like mathematics, as do teachers.

- Pupil attitude toward mathematics is related to intelligence and achievement.

- Pupils have more positive attitudes toward arithmetic when it is taught as a useful skill, with practical values for out-of-school situations.

- Socioeconomic level affects background and achievement, but not so much in mathematics as in other curricular areas.

- Increased parent knowledge of classroom mathematics activities results in higher pupil mathematical achievement.

- The teacher and the strategies he uses are important.

- Teacher background is related to pupil achievement.

- The mathematical competency of teachers is inadequate but seems to be improving.

Teaching Methods and Strategies

- The teaching methods which are used can decrease the difficulty of the learning task.
- Meaningful teaching is better than mechanical, rote teaching.
- Inductive discovery strategies are effective, especially for retention and transfer.
- Diagnosis of pupil errors can be done effectively by listening to pupils verbalize while working.
- The decomposition method of subtraction may be better than the equal additions method for developing understanding.
- Programed instruction can be used to present many topics effectively.

IPRIP

No. 11-C

Current Research on Elementary Mathematics

The following documents will enable school administrators and teachers to stay abreast of some of the latest developments in the teaching of elementary mathematics. Copies of these documents are available either in microfiche (MF) or hard copy (HC) from the ERIC Document Reproduction Service, The National Cash Register Company, 4936 Fairmont Avenue, Bethesda, Maryland 20014 at the prices indicated below with each document.

Mathematics Program, K-8, 1967-1968 *Strands Report*, Part 2. ED 022 676. 84 p. MF - 50¢; HC - $4.20

The Use of Games to Facilitate the Learning of Basic Number Concepts in Preschool Educable Mentally Retarded Children. ED 023 243. 91 p. MF - 50¢; HC - $4.65.

Bibliography of Research Studies in Elementary School and Pre-School Mathematics. ED 023 464. 55 p. MF - 25¢; HC - $2.85.

Pre-Service and In-Service Education in Mathematics of Colorado Elementary School Teachers--A Status Report. ED 023 576. 6 p. MF - 25¢; HC - 40¢.

Comparison of Two Teaching Techniques in Elementary School Mathematics. ED 023 595. 53 p. MF - 25¢; HC - $2.75.

Guidelines for Mathematics in the Elementary School. ED 026 237. 32 p. MF - 25¢; HC - $1.70.

An Elementary Mathematics Program. ED 031 397. 13 p. MF - 25¢; HC - 75¢.

Evaluation of Three Computer-Assisted Instruction Programs. ED 031 408. 62 p. MF - 50¢; HC - $3.20.

Papers for the Research Reporting Sections of the 47th Annual Meeting of the National Council of Teachers of Mathematics. ED 029 790. 107 p. MF - 50¢; HC - $5.45.